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# Preliminary study on the relationship between dry density of sands and the grading entropy parameters

## Etude préliminaire sur la relation entre la densité sèche des sables et les paramètres d'entropie de gradation

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**ABSTRACT:** Some earlier data were analysed, searching the relation between the minimum dry density $e_{max}$ and the two grading entropy parameters (mean log diameter and fraction number characteristic). The data were split into two components. The first data component – constituting the major part of the density – was the linear function of the mean log diameter, the second one followed the shape of the entropy diagram and it was about the same for the constant vale of the mean log diameter. The density– in terms of normalized the mean log diameter– was maximal around at the point where internal structure changed.

**RÉSUMÉ:** Certaines données antérieures ont été analysées, en recherchant la relation entre l'emax à densité sèche minimale et les deux paramètres d'entropie. Les données ont été divisées en deux composantes. La première composante de données – constituant la majeure partie de la densité – était la fonction linéaire du diamètre moyen de la bûche, la deuxième suivait la forme du diagramme d'entropie et elle était à peu près la même pour la Vale constante du diamètre moyen des grumes. La densité – en termes de normalisation du diamètre moyen des grumes – était maximale autour du point où la structure interne changeait.

**Keywords:** Grading entropy, interpolation, sands, minimum dry density $e_{max}$

# 1 INTRODUCTION

In the pioneering research work on sand density of Lorincz (1986, 1990), the artificial mixture series of natural sand grains with increasing, mean log diameters, various fraction numbers $N$=1 to 5 were tested for minimum dry density $e_{max}$. The grading curve series were 'optimal' (fractal) or gap-graded.

The data were used for interpolation (Lorincz, (1986, Imre et al, 2014). and, to explore the relation between density and grading entropy. According to the conclusion of Lorincz, the minimum dry density $e_{\max}$ was a maximum value at around $A$=2/3 for each series. The global maximum occurred for gapgraded mixtures.

Analysing the $e_{max}$ data further, a split (into weighted mean fraction density and density increment, Lorincz 1986) was applied. Some near antisymmetric and near symmetric-like relations were found for these in terms of the grading entropy parameters (Imre et al, 2015).

In this paper the antisymmetric and symmetric-nature of the split is proved with statistical mathematical tools (model fitting) amd some data measured at Ruhr University of Bochum are presented. The "$A$~2/3 maximum density conjecture" is supported by both. The data analysis results in the statement that for constant $A$ the density is a near-linear function of the relative frequencies. Using this, linear optimisation theory may explains why the maximum/minimum appear at the boundaries of the constant $A$ domains.

# 2 GRADING ENTROPY

## 2.1 *The space of grading curves*

The relative frequencies of the fractions $x_i$ ($i$ = 1, 2, ..$N$) for each grading curve fulfil:

$$\sum_{i=1}^{N} x_i = 1, \quad x_i \geq 0, \; N \geq 1. \tag{1}$$

where $N$ is the number of the fractions between the finest and coarsest non-zero fractions:

$$N = j_{\max} - j_{\min} + 1 \tag{2}$$

The relative frequencies $x_i$ - and the space of grading curves with $N$ fractions - can be identified with the barycentre coordinates in an $N$-1 dimensional simplex (see Figs 2. to 3.).

For a fixed $N$, both the non-normalized entropy map $[\Delta,N] \rightarrow [S_0,\Delta S]$ and, the normalized entropy map $[\Delta,N] \rightarrow [A,B]$ are continuous on the open simplex and can continuously be extended to the closed simplex.

## 2.2 *Derivation of the grading entropy*

Two statistical cell systems are used. The fractions are defined by successive multiplication with a factor of 2, starting from an arbitrary $d_0$ as follows ($j$ =1, 2.., Table 1.).

$$2^{j} d_0 \geq d > 2^{j-1} d_0 \tag{3}$$

where fractions are numbered by $j$ (serial number). A uniform cell system with $d_0$ width is used assuming that the distribution within a fraction is uniform. The base of the logarithm is set into 2 in statistical entropy Equation:

$$s = -\frac{1}{ln 2} \sum_{i=1}^{m} \alpha_i \, ln \alpha_i \tag{4}$$

so that the maximal value is equal to 1 where the relative frequencies of the two cells are equal.

The number of the elementary cells $C_i$ in the fraction $i$ is equal to:

$$C_i = \frac{2^i d_0 - 2^{i-1} d_0}{d_0} = 2^{i-1} \tag{5}$$

The relative frequency of any elementary cell in fraction $i$ is equal to:

$$\alpha_i = \frac{x_i}{C_i} \tag{6}$$

where $x_i$ is the relative frequency of fraction $i$. $C_i$ is the number of the elementary cells in fraction $i$, and $x_i$ is the relative frequency of fraction $i$.

Table 1. Definition and properties of fraction *j*.

| *j* | 1 | | 23 | 24 |
|---|---|---|---|---|
| Limits | 1 to 2 | | $2^{22}$ to $2^{23}$ | $2^{23}$ to $2^{24}$ |
| $S_{0j}$ [-] | 1 | | 23 | 24 |

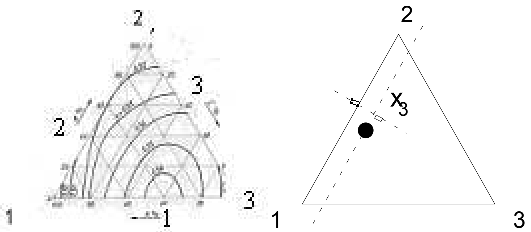


*Figure 1. (left) Representation of the 3- fraction soils with the interpolated minimum dry density lines (with maximum at the gap-graded edge, Lorincz, 1986). (right) The relative frequency $x_N$ of a given simplex point is determined by the distance of the face 1…N-1 and its parallel transportgiven simple point.*

The grading entropy $S$ is derived by inserting the relative frequency of the secondary cell $\alpha_i$ into the statistical entropy Equation of finite dscrete distributions (Equation (4)):

$$S = -\frac{1}{\ln 2} \sum_{x_i \neq 0} C_i \frac{x_i}{C_i} \ln \frac{x_i}{C_i}, x_i \geq 0 \qquad (7)$$

The grading entropy $S$ is split into the base entropy $S_o$ and the entropy increment $\Delta S$:

$$S = S_0 + \Delta S \qquad (8)$$

The base entropy $S_0$ and the normalized form $A$:

$$S_0 = \sum x_i S_{0i} = \sum x_i i \text{ and } A = \frac{S_o - S_{o\min}}{S_{o\max} - S_{o\min}} \qquad (9)$$

where $S_{0k}$ is the $k$-th fraction entropy (Table 1), $S_{0max}$ and $S_{0min}$ are the entropies of largest and smallest fractions, respectively. The entropy increment $\Delta S$ and the normalized version $B$:

$$\Delta S = -\frac{1}{\ln 2} \sum_{x_i \neq 0} x_i \ln x_i . \; B = \frac{\Delta S}{\ln N} \qquad (10)$$

where $S_{0i}$ is the grading entropy of the $k$-th fraction, which is defined as follows (Table 1.):

$$S_{ok} = \frac{\ln C_k}{\ln 2} \;\; S_{ok} = k - 1 \qquad (11)$$

The grading entropy parameters induces a secondary structure on the space of the grading curves. The A = const. condition defines parallel *N*-2 dimensional hyper-plane sections of the *N*-1 dimensional simplex, , the A = const., B=const. condition defines *N*-3 dimensional topological circles (Figs 2. to 3.).

## 2.3 Entropy diagrams

### 2.3.1 The lines of the entropy diagrams

The image of the compact simplex is compact (Fig 4. left), having a maximum and a minimum value for every possible *A* or $S_0$. These coincide for N=2 being equal to the specific entropy function of Eq (4). This line approximately determines all lines of the entropy diagrams in terms a constant multiplyer. The optimal point at maximum *B* for a fixed *N* and *A*:

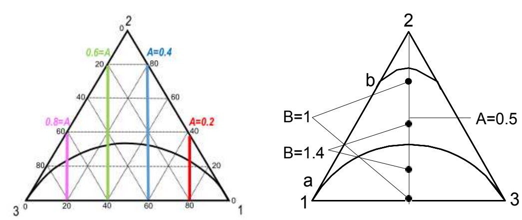


*Figure 2. N=3. (left) Points with fixed A (N-2=1 dimensional planes) with the optimal line. (right): Points with fixed A and B.*

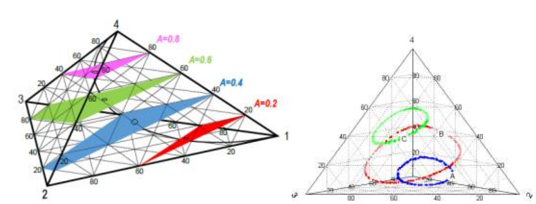


*Figure 3. N=4. (left) Points with fixed A (N-2=1 dimensional planes) with the optimal line. (right): Points with fixed A and B (N-3=1 dimensional circles). (A=0.66 B=1.2,: A=0.5 B=1.2,:*

*Table 2. The A coordinates for the maximum entropy point (at a fixed fractal dimension n = 2)*

| **N** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|
| *A* | 0,67 | 0,71 | 0,75 | 0,79 | 0,82 | 0,84 |

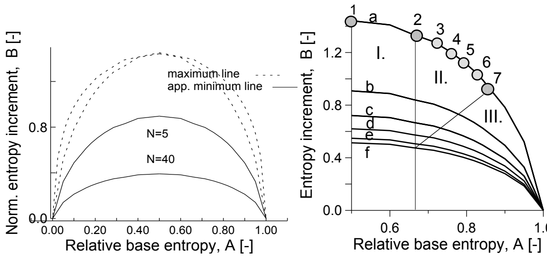


*Figure 4. (left) The maximum and the minimum lines of the normalized entropy diagrams, depending on N. The maximum lines related to N=2 and 40 coinciding at A=0.5, at the symmetry axis and slightly differing for other A values. (right) The internal stability domains for N=7 are: I: piping, II: transition, III: stable. The points 2 to 7 are the maximum entropy points for N=2 to 7, resp. lines as b to f the approximate minimum lines for N =2 to 7.*

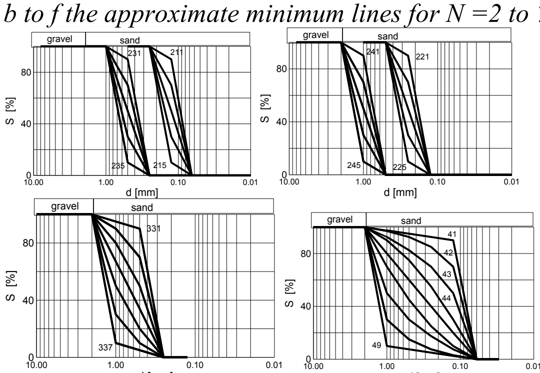


*Figure 5. Some optimal grading curves of Lorincz*

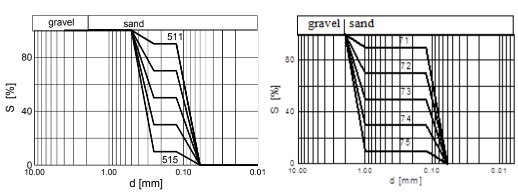


*Figure 6. Some gap-graded grading curve series of Lorincz.*

$$x_1 = \frac{1}{\sum_{j=1}^{N} a^{j-1}} = \frac{1-a}{1-a^N},\ x_j = x_1\ a^{j-1} \quad (12)$$

where parameter $a$ is the root of the equation :

$$y = \sum_{j=1}^{N} a^{j-1}[j-1-A(N-1)] = 0. \quad (13)$$

The optimal grading curves have fractal distribution, the fractal dimension n is as follows.

$$n = 3 - \frac{\log a}{\log 2} \quad (14)$$

While $a$ varies from 0 to 1 or 1 to $\infty$, $A$ varies from 0 to 0.5 or from 0.5 to 1, $n$ varies from $\infty$ to 3 or from 3 to $-\infty$, respectively. The maximum entropy points are at a=n=2 (Figure 4, Table 2).

### *2.3.2 The internal stability rule*

The internal stability rule of the grading entropy theory (Figure 4 right) is defined by vertical flow tests (Lorincz, 1986). For A < 2/3, the mixtures are internally unstable, and stable in zone III. . The transitional zone depends on *N* and the maximum entropy point (Table 2).

## 3 METHODS

### *3.1 Density measuremens*

The minimum dry density $e_{max}$ or $s_{\min}$ test were originally made on five fractions and artificial - mixtures of natural sand grains with optimal or gapgraded soils using Proctor mold (see Figures 5, 6). The fractions were: 0.06-0,125mm, 0.125-0.25 mm, 0.25-0.5 mm, 0.5-1 mm, 1-2 mm. The experimental tests have been started to be repeated and extended at Bochum University. More than 5 fractions were considered and, to determine the maximum point, additional series were included. The DIN mold was used.

### *3.2 Density variables*

The void ratio *e*, solid volume ratio *s* or its inverse, the specific volume *v*:

$$s = \frac{V_S}{V} = \frac{1}{v} = \frac{1}{1+e} \quad (15)$$

Concerning the grading entropy parameters, the base entropy $S_0$ was considered as a mean abstract log diameter $i_0$ between $S_{0max}$ and $S_{0min.}$. The relative base entropy A was considered as a normalised abstract mean log diameter $k_m$ between 0 and 1 and were used in the representation.

The entropy increment $\Delta S$ was used since it measures the 'disorder'due to the mixing of the fractons. It is maximal if all relative frequency is the same. The measured solid volume ratio $s$ was split into two parts for every mixture:

$$s_0 = \sum_{i=i_{\min}}^{i_{\max}} x_i s_i \,, \tag{16}$$

$$\Delta s = s_{\min} - s_0 \tag{17}$$

using the $x_i$ relative frequencies and the measured solid volume ratio $s_i$ for fractions where $s_i$ refers the $s_{min}$ of fraction $i$.

### 3.3 Analyses with model fitting

A linear line was fitted on all the $S_0$ - $s_0$ relation:

$$s_0 = cS_0 \tag{18}$$

For mixtures, the fully symmetric $\Delta S$ line of the 2-fraction mixtures, denoted s $\Delta S_2$, with maximum of 1 (Eq 4), was fitted to the $S_0$ – ($s_{\min}$- $s_0$) data:

$$s_{\min} - s_0 = k\Delta S_2 \tag{19}$$

where $k$ was identified. This fit was simultaneously made for the same type of mixtures for a fixed N(eg., N=2, continuous) to verify the symmetry assumption.

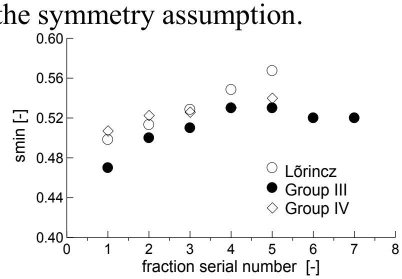


*Figure 7. The fraction data, measured by Lorincz 1986 and, in Bochum (Groups III and IV).*

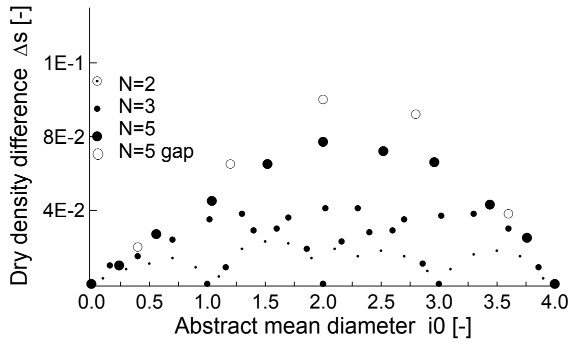


*Figure 8. The computed $s_{min}$ - $s_0$ values, with a feature of the symmetry. (b) The optimal mixtures, indicating the N=5 gap-graded.*

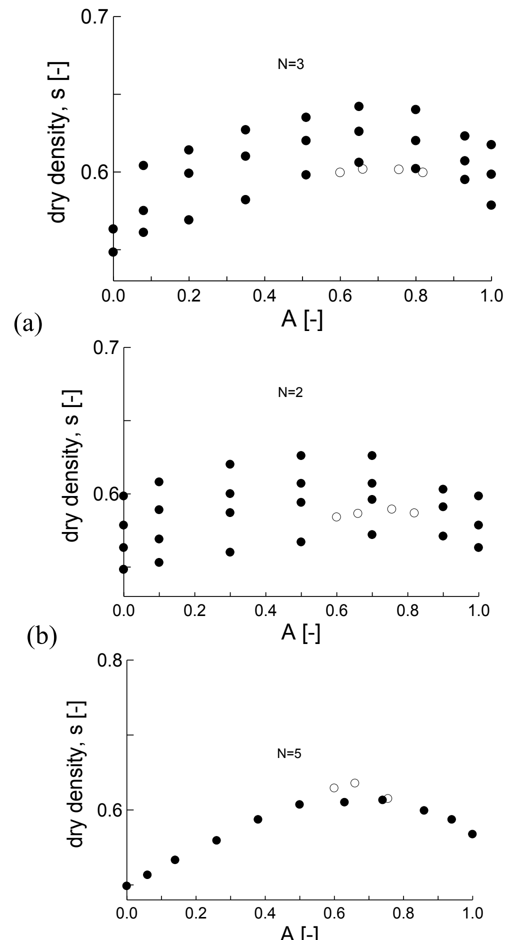


*Figure 9. (a) to (c) Measured optimal data of Lorincz: full symbols, Bochum data: open symbols.*

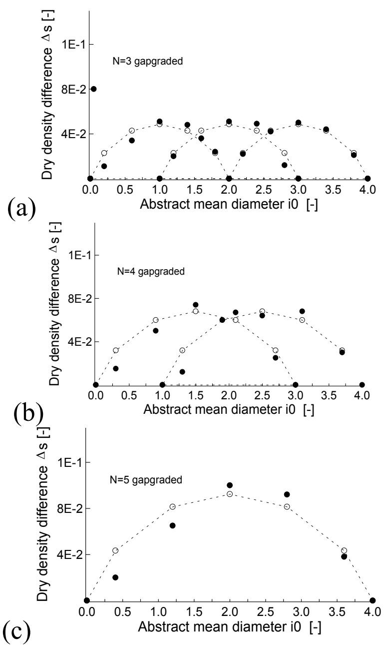


*Figure 10. Model fitting. The best-fit grading entropy increment line, gap-graded case (a) to (c) N=3, 4, 5.*

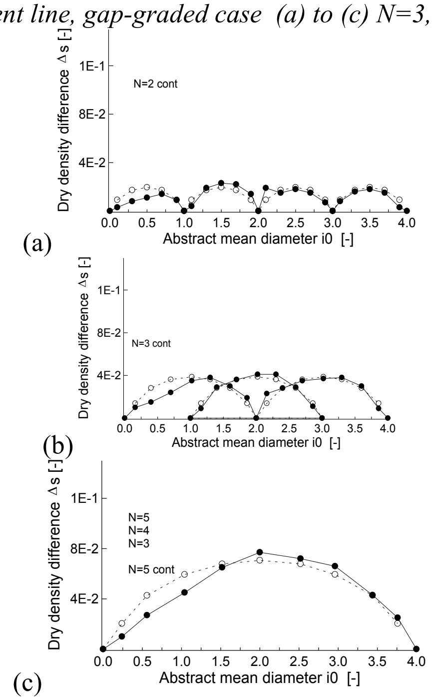


*Figure 11. Model fitting. The difference $s_{min}$ - $s_0$ and the best-fit grading entropy increment line, gapgraded mixtures. (a) to (c) N=2,3, 5*

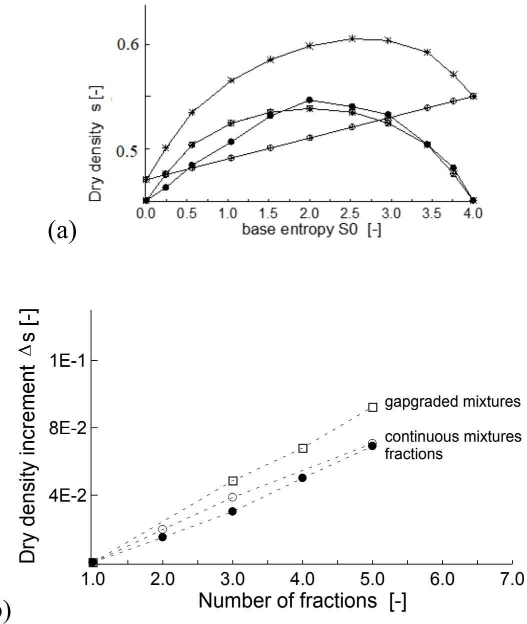


*Figure 12. (a) Sum of the components for the N=5 continuous mixtures, explaining the maximum location. (b) The $s_{min}$-$s_0$ at A=0.5, with the increment values of the fraction densities $s_{min,i}$ -$s_{min,1}$.*

*Table 3. Continuous mixtures. identified k values*

| ***N*** | **2** | **3** | **5** |
|---|---|---|---|
| *k* | 0,02 | 0,04 | 0,07 |

*Table 4. Gap-graded mixtures. identified k values*

| ***N*** | **4** | **5** |
|---|---|---|
| *k* | 0,07 | 0,09 |

*Table 5. The max/min for $s_{min}$, gapgraded soils, N=5*

| ***N*** | **5** | **4** | **3** |
|---|---|---|---|
| *max/min* | 1,28 | 1,20 | 1,14 |

*Table 6. The max/min for $s_{min}$, optimal soils, N=5*

| ***N*** | **5** | **3** | **2** |
|---|---|---|---|
| *max/min* | 1,23 | 1,12 | 1,05 |

*Table 7. Minimum, mean, maximum $s_{min}$, N=5*

| ***A*** | **0.25** | **0.5** | **0.75** |
|---|---|---|---|
| edge 1-2 or 3-4 | 0,51 | 0,53 | 0,55 |
| optimal point | 0,56 | 0,61 | 0,61 |
| edge 1-N | 0,58 | 0,63 | 0,64 |

*Table 8. -Minimum, mean, maximum $s_{min}$, N=3*

| *A* | **0.25** | **0.5** | **0.75** |
|---|---|---|---|
| edge 1-2 or 2-3 | 0,52 | 0,52 | 0,52 |
| optimal point | 0,52 | 0,55 | 0,54 |
| edge 1-N | 0,54 | 0,56 | 0,57 |

## 4 RESULTS

The measured results are shown in Figures 7 to 9, Tables 3 to 4. The $s_{min}$ for soil fractions increased with the fraction serial number *j* for Lorincz (1986) but the increase was only effective for the smallest few fractions in the Bochum data (the differences due to the testing mode are treated in Imre et al 2011 and 2014).

According to Figure 9, the $s_{min}$ for the mixture series had a maximum for both the optimal or gapgraded series at around *A*=2/3 being the greater for gapgraded soils.

The split data and the results of the model fitting are shown in Figures 10 to 11. Using fraction data, *c* was identified as 0,016. For mixtures, the $\Delta S$ line of the 2-fraction mixtures, denoted as $\Delta S_2$, with maximum of 1, was fitted to the $S_0 - (s_{\min} - s_0)$ data. The same type of mixtures with same *N* (see Figures 5, 6 ) were simultaneously fitted.

The results are summarized in Figure 12. The sum of the fitted functions had a maximum at around A=2/3. The identified *k* values - the maximal density increments at *A*=0,5 - increased with *N,* they plot into near linear lines with slightly smaller slope for the continuous than for the gap-graded, being both slightly steeper than the fractions line with slope of 0,016.

## 5 DISCUSSION

### 5.1 The density variation in the simplex

The maximum and the minimum value of $s_{min}$ within the mixture series for fixed *N* may differ by a factor of 1.2, as shown in Tables 5 and 6. The maximum for optimal or gapgraded series was at around *A*=2/3 being the greatest for gapgraded soils. This results imply the following assumption. The $s_{min}$ for fixed *A* and *N* is controlled by the quantity of the largest fraction (see the verticals of Figure 1 left, Imre et al, 2009). These spatial distribution results and assumptions can be explained by linear optimisation theory as follows.

The $s=s_{min}$ function can be rewritten as follows in the suggested split model:

$$s_0 = \sum_{i=i_{\min}}^{i_{\max}} x_i s_i \,, \quad \Delta s = s_{\min} - s_0 \tag{20}$$

$$s = \sum_{i=i_{\min}}^{i_{\max}} x_i c_i + N f(A) \tag{21}$$

Since $s_i$ dry density of the fractions measured by Lorincz increases with i, $c_1 < c_{2\ldots} < c_N$, the second term is less than about 1/5 of the first term and can be considered about constant for fixed *A*. It follows that the $s=s_{min}$ function is nearly linear for fixed *A,* therefore, the extreme values of the density are encountered at the boundaries, the mean at the optimal point.

Being the largest coeffiicient the $c_N$, following from the geometrical meaning of relative frequency $x_N$ and the *A*=const hyperplanes (Figures 2, 3) it follows from that the extreme density values are determined by the maximum/minimum value of $x_N$. If the s function is basically linear, its conditional maximum or minimum will appear on each the *A*=const hyperplane at edge 1-N or the edge with minimum $x_N$, resp.

### 5.2 The density increment in compaction

The mixture $s_{min}$ value is dependent on *A*, with ratio of the minimum and maximum $s_{min}$ between 1.05 to 1.28 (Tables 5 and 6).

The ratio of the minimum and the maximum dry density $s_{min} / s_{max}$ is comparable with this, being equal to eg., 0.822 (=1/1.2) for calibration sand data (Imre et al, 2011, Kabai, 1968).

# 6 CONCLUSIONS

## 6.1 Fraction results

According to the measurements of Lorincz, the density of the fractions $s_{min,i}$ (sometimes denoted by $s_i$) increases linearly with $S_o$ in the tested diameter ranges, where $S_o$ for the i-th fraction is equal to $S_{oi}$. This result was not fully reproduced possibly due to the different testing and soil conditions.

## 6.2 Grading curves with varying A

(1) According to the measurements Lorincz, the mixture density $s_{min}$ (sometimes denoted by $s$) in terms of $A$ has a maximum around $A$=2/3 for each tested grading curve series where $A$ varied. This was reproduced by the repeated tests.
(2) In the suggested split model, the $s_0$ (mean fraction density) is substracted from the mixture density. The remaining increment $s_{min}$ -$s_0$ is zero for the fractions and, is positive and controlled by the entropy increment $\Delta S(A)$ for the mixtures such that the maximum of $s_{min}$ -$s_0$ at $A$=0.5 increases with $N$ about linearly, similarly to the fraction density increase with the fraction serial number. The $s_0$ (mean fraction density) in terms of $S_o$ follow the same linear relation as the fraction density $s_{min,i}$ in terms of $S_o$.
(3) The split model may explain why the mixture density $s_{min}$ in terms of $A$ has a maximum around $A$=2/3.

## 6.3 Grading curves with constant A

For a fixed constant $A$ section (shown eg., in Figures 2 to 3), according to the observations, the $s_{\min}$ is dependent on the quantity of the largest fraction. The optimal point has a mean value, the gap-graded point a maximum value, the simplex edge point where the largest fraction is missing, the minimum. This result can be explained by the fact that the dry density function is basically linear. Due to linear optimisation rules, on the constant $A$ sections of the grading curve space, the minimum and the maximum of $s_0$ is occuring at the boundaries where $x_N$ is a maximum or a minimum.

It can be noted that all result can be explained by finite geometry reasons. Larger diameter ranges (larger variety of grain sizes) may result in denser packing both for fractions and mixtures. It can be noted that the $A$=2/3 for limit is an internal stability limit which can be explained by microstructural reasons as follows. The $A$=2/3 limit determines a boundary where ‘coarse in fine’ structure changes into ‘fine in coarse’ configuration (see eg., Goudarzy, 2015).

# 7 REFERENCES

Goudarzy M 2015. Micro and Macro Mechanical Assessment of Small - Intermediate Strain Properties of Granular Material Ph D Thesis, Faculty of Civil Engineering, U Bochum.

Imre, E; Lorincz, J; Trang, Q.P; Fityus, S. Pusztai, J; Telekes, G; Schanz, T. 2009. Some dry density transfer function for sands. KSCE J of C Eng 13(4): 257-272.

Imre E, Fityus S, Keszeyné E, Schanz T 2011. A Comment on te Ratio of the Maximum and Minimum Dry Density for Sands. Geotechnical Engineering 42(4) pp. 77-82.

Imre E , Lorincz J , Hazay M , Juhász M , Rajkai K , Schanz T , Lins Y , Singh V P , Hortobagyi Zs. 2014. Sand mixture density In: of UNSAT2014. pp. 691-697.

Imre E, G Kecskés G, K Rajkai K, T Schanz k, Zs Hortobágyi Zs, V P Singh V P, S Fityus S. 2015. Sand behaviour in terms of the grading curve. XVI ECSMGE Edinburgh, 3753-3758.

Kabai, I. 1968. The compactibility of sands and sandy gravels. Acta Technica H, 113-124.

Lorincz, J 1986. Grading entropy of soils Doctoral Thesis, BME, Budapest.

Lorincz, J 1990. Relationship between grading entropy and dry bulk density of granular soils Periodica Polytechnica 34:3:255-265.