ON THE ALGEBRAIC FOUNDATIONS OF THE VECTOR ϵ-ALGORITHM

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Abstract. We review the Clifford algebraic foundations of versions of the vector ϵ-algorithm. This involves the formation of rational approximants to vector-valued functions defined by a power series. We summarise their properties and demonstrate how a study of these algebraic constructs leads to convergence results concerning the vector ϵ-table which we apply to the iterative solution of simultaneous linear equations. The generalisation of the ϵ-algorithm to vector rational Hermite interpolants is also presented. Finally, we consider various algebraic representations for generalised inverse rational approximants and interpolants.

Key words: vector ϵ-algorithm, vector rational approximants, Hermite interpolants, iterative solution to linear equations.

1. Introduction

There are many problems in Science and Engineering which are tackled using numerical techniques based on iterations — e.g., systems of linear or non-linear algebraic equations arising from the discretisation of partial differential equations. These methods produce a sequence of vectors whose limit furnishes the desired solution or approximation. However, the numerical convergence may be too slow for practical purposes and resort made to methods for accelerating the convergence of the vector sequence. The more common of these methods are classed as either of polynomial or rational type — Smith *et al.* (1987) provide a review of many of the corresponding algorithms. The vector ϵ-algorithm, first introduced by Wynn (1962), is often used to implement methods of the rational kind.

In this paper, we are concerned with the algebraic theory underlying the vector ϵ-algorithm. This theory is based on the use of real and complex Clifford algebras. There is another approach to establishing a theoretical grounding for this algorithm proposed by Graves-Morris (1983), which we touch upon later and link to our algebraic view. The advantage of the former perspective is that we produce a framework for discussing the rational approximation to vector data whose structure parallels that for the usual scalar theory — e.g., this includes continued fractions and three-term recurrence relations. One of the motivations for this is the hope that proofs of theorems and extensions of algorithms valid in the usual theory may be adapted for the vector case.

We introduce the ϵ-algorithm by considering a given sequence of vectors $\{\mathbf{s}_j\}$

with $\mathbf{s}_j \in \mathbb{C}^n$, $j = 0, 1, 2, \ldots$. Initialisation is provided by the following definitions:

$$\epsilon_{-1}^{(j+1)} := 0 \quad \text{and} \quad \epsilon_0^{(j)} := \mathbf{s}_j, \quad j = 0, 1, 2, \ldots. \tag{1.1}$$

Subsequent iterations are performed using the rhombus rule,

$$\epsilon_{k+1}^{(j)} := \epsilon_{k-1}^{(j+1)} + \left[\epsilon_k^{(j+1)} - \epsilon_k^{(j)}\right]^{-1} \tag{1.2}$$

with $j = 1, 2, 3, \ldots$ and $k = 0, 1, 2, \ldots$, which connects the four elements shown in Fig.1.

$$\begin{array}{ccc} & \boxed{\text{U}} & \\ \boxed{\text{L}} & & \boxed{\text{R}} \\ & \boxed{\text{D}} & \end{array} \equiv \begin{array}{ccc} & \epsilon_k^{(j)} & \\ \epsilon_{k-1}^{(j+1)} & & \epsilon_{k+1}^{(j)} \\ & \epsilon_k^{(j+1)} & \end{array}$$

$$\boxed{\text{R}} = \boxed{\text{L}} + \left[\ \boxed{\text{D}} - \boxed{\text{U}}\ \right]^{-1}$$

Fig. 1. Rhombus Rule

As in the scalar case — see e.g., Baker and Graves-Morris (1981) — (1.2) is used to construct, column by column, the two-dimensional table illustrated in Fig. 2.

$$\begin{array}{ccccccc} & \epsilon_0^{(0)} & & & & & \\ \epsilon_{-1}^{(1)} & & \epsilon_1^{(0)} & & & & \\ & \epsilon_0^{(1)} & & \epsilon_2^{(0)} & & & \\ \epsilon_{-1}^{(2)} & & \epsilon_1^{(1)} & & \cdot & & \\ & \epsilon_0^{(2)} & & \epsilon_2^{(1)} & & \cdot & \\ \epsilon_{-1}^{(3)} & & \epsilon_1^{(2)} & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & & \cdot \end{array}$$

Fig. 2. The Vector ϵ-table

However, in order to implement (1.2) we require a definition of the inverse of a vector.

2. Historical Comment

In his early work, Wynn (1962) suggests a vector inverse based on the Moore-Penrose generalised inverse; viz.,

$$\mathbf{v}^{-1} := \frac{\mathbf{v}^*}{\mathbf{v} \cdot \mathbf{v}^*}, \quad \mathbf{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{C}^n \tag{2.1}$$

where * denotes complex conjugation. With this definition (1.2) reduces to the scalar algorithm for $n = 1$. This proved to be a success when the method was applied to the numerical solution of systems of linear or non-linear algebraic equations using iterative techniques — see e.g., Smith *et al.* (1987), Wynn (1962), Brezinski and Rieu (1974), Gekeler (1972). More recent work by Graves-Morris (1992) and by da Cunha and Hopkins (1994) has shown its advantage, and its limitations, in accelerating SOR (Successive Overrelaxation) solutions to large systems of linear equations.

Using (2.1) it was observed that the even columns of the ϵ-table usually converged faster the further they were to the right — reflecting the behaviour of the scalar ϵ-table. In fact, for the latter there is a theory based on rational approximation of functions, which leads to results governing the behaviour of the even columns. In order to construct a vector analogue we need to consider the definition and convergence properties of rational approximants to vector-valued functions whose power series are known:

$$\mathbf{f}(z) = \mathbf{c}_0 + z\mathbf{c}_1 + z^2\mathbf{c}_2 + \dots, \quad z \in \mathbb{C} \tag{2.2}$$

where $\mathbf{c}_0 := \mathbf{s}_0$ and $\mathbf{c}_j := \mathbf{s}_j - \mathbf{s}_{j-1}$, $j = 1, 2, \dots$.

However, there was no fundamental theory available to Wynn to substantiate this approach of using the vector ϵ-algorithm as something more than an “adventitious computational technique” (Wynn, 1968). In particular, there was no convergence theory. Indeed, other inverses were suggested by Brezinski (1975, 1991).

In order to remedy this lack of a theoretical basis Wynn (1963) pursued an investigation of continued fractions (i.e., rational approximants and interpolants) whose elements were taken from a non-commutative, associative algebra. This led to a development along lines similar to that of the usual scalar constructs, without employing determinants which played a useful role in the theory of the latter — see e.g., (Baker and Graves-Morris, 1981). The application of (1.2) to vector data was attempted by exploiting a representation of vectors by Clifford numbers, i.e., a 1–1 correspondence between vectors $\mathbf{v} \in \mathbb{C}^n$ and square matrices V such that:

$$\begin{array}{rcl} \mathbf{v} & \leftrightarrow & V \\ \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 & \leftrightarrow & \alpha V_1 + \beta V_2 \\ \mathbf{v}^{-1} & \leftrightarrow & V^{-1} \end{array} \tag{2.3}$$

where $\mathbf{v}^{-1}$ is defined by (2.1). (See discussion in Section 5.) However, it was soon recognised that α and β had to be restricted to the reals. With this constraint, McLeod was able to use the results of Wynn (1963) to prove the following theorem for the case of the $\beta_i \in \mathbb{R}$, $i = 0, 1, \dots, K$:

Theorem 2.1 (McLeod 1971, Graves-Morris 1983) If the vector sequence $\{\mathbf{s}_j\}$, with $\mathbf{s}_j \in \mathbb{C}^n$, $j = 0, 1, 2, \dots$ satisfies a non-trivial recurrence relation

$$\sum_{i=0}^{K} \beta_i \mathbf{s}_{i+j} = \left[\sum_{i=0}^{K} \beta_i\right] \mathbf{a}, \quad j = 0, 1, 2, \dots \tag{2.4}$$

with $\beta_i \in \mathbb{C}$ such that $\beta_0\beta_K \neq 0$, then the vector ϵ-algorithm using (2.1) leads to

$$\epsilon_{2K}^{(j)} = \mathbf{a} \tag{2.5}$$

provided that zero divisions are not encountered in the construction. $\Box$

The recurrence relation (2.4) is exactly of the form encountered in the iterative solution to systems of linear equations. The restriction mentioned above also ruled out the representation in this algebra of vector-valued rational interpolants with complex data points — see Section 5.

In the 1980's a different tack was adopted by Graves-Morris who invented and investigated generalised inverse rational forms, $\mathbf{P}(z)/Q(z)$ — both as approximants and as interpolants. Such a form is of type $\{m/2k\}$ if $Q(z)$ is a real analytic polynomial of even degree $2k$ in general, and each $P_i(z)$, $i = 1, \ldots, n$ is a polynomial of maximum degree m. As their name suggests they are based on the vector inverse (2.1), which is reflected in the division property

$$Q(z)|\mathbf{P}(z) \cdot \mathbf{P}^*(z) \tag{2.6}$$

where $\mathbf{P}^*(z)$ denotes the functional complex conjugate of $\mathbf{P}(z)$ defined by Graves-Morris and Jenkins (1986). For the particular form known as a Generalised Inverse Padé Approximant, the following order condition is satisfied

$$\frac{\mathbf{P}(z)}{Q(z)} - \mathbf{f}(z) = O(z^m), \tag{2.7}$$

which is different from the usual Padé theory. However, in 1986, Graves-Morris with Jenkins published a proof of McLeod's theorem for $\beta_i \in \mathbb{C}$, and subsequently, with Saff in 1988 and 1991, two theorems governing the convergence of the even columns of the vector ϵ-table for certain types of functions.

Nonetheless, Clifford algebras played no role in the axiomatic development of this theory — a theory whose results were similar to those for scalars, but whose underlying structure did not reflect that of the usual theory — for example, there were no recurrence relations for continued fractions and the properties of the Generalised Inverse Padé Approximants did not follow the normal version. In the next section we describe a new approach which does not suffer from these drawbacks.

3. Vector Padé Approximants

3.1. Definition

We use the complex Clifford algebra $C\ell(\mathbb{C}^n)$ to represent vectors $\mathbf{v} \in \mathbb{C}^n$ by setting

$$\mathbf{v} = \sum_{i=1}^{n} v_i e_i \tag{3.1}$$

where e_i, $i = 1, 2, \ldots, n$ generate the algebra with $e_i^2 = e_0$ — see e.g., Rasevskii (1957) and Porteous (1981). Then the inverse of a vector is given by

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \tag{3.2}$$

where $\mathbf{v} \cdot \mathbf{v}$ denotes the usual scalar product. This agrees with the generalised inverse (2.1) for $\mathbf{v} \in \mathbb{R}^n$.

Although any product of vectors is well-defined in $C\ell(\mathbb{C}^n)$, there are some which yield simple results: e.g., if $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n \subset C\ell(\mathbb{C}^n)$ then

$$\mathbf{uv} + \mathbf{vu} = 2\mathbf{u} \cdot \mathbf{v} e_0 \tag{3.3}$$

$$\mathbf{uvu} = 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - \mathbf{v}(\mathbf{u} \cdot \mathbf{u}) \tag{3.4}$$

— in which e_0 is the algebra identity.

We now consider a vector-valued function, $\mathbf{f} : \mathbb{C} \to \mathbb{C}^n$, which has a MacLaurin series

$$\mathbf{f}(z) = \mathbf{c}_0 + z\mathbf{c}_1 + z^2\mathbf{c}_2 + \ldots, \quad z \in \mathbb{C}, \quad \mathbf{c}_i \in \mathbb{C}^n, \quad i = 0, 1, \ldots$$

(see eq. (2.2) above) valid in some neighbourhood of the origin. The right-handed $[L/M]$ Vector Padé Approximant (VPA) to $\mathbf{f}(z)$, if it exists, is defined by

$$[L/M](z) := p(z)[q(z)]^{-1}$$

for which

$$[L/M](z) - \mathbf{f}(z) = O(z^{L+M+1}) \tag{3.5}$$

where $p(z)$ and $q(z)$ are polynomials in $z \in \mathbb{C}$ over $C\ell(\mathbb{C}^n)$ of maximum degrees L and M respectively. The left-handed version is obtained by applying the reverse anti-automorphism denoted here by $\tilde{}$. The two versions can be shown to be identical — see (Roberts, 1993). This is sufficient to guarantee the uniqueness of the $[L/M]$ VPA to $\mathbf{f}(z)$. These approximants may be arrayed in a two-dimensional table as in Fig.3. We refer the interested reader to Roberts (1990) for a discussion of the

$$\begin{array}{cccc} [0/0] & [1/0] & [2/0] & \ldots \\ [0/1] & [1/1] & [2/1] & \ldots \\ [0/2] & [1/2] & [2/2] & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Fig. 3. The Vector Padé Table

properties of VPA's which reflect those of the usual Padé approximants.

3.2. Construction

If we adopt the Baker convention by setting $q(0) := e_0$, then, on multiplying (3.5) by $q(z)$ from the right we obtain

$$p(z) - \mathbf{f}(z)q(z) = O(z^{L+M+1}). \tag{3.6}$$

This yields a system of $(L + M + 1)$ linear equations in the $(L + M + 1)$ unknown coefficients of $p(z)$ and $q(z)$. As a simple illustration we consider the $[1/1]$ VPA to $\mathbf{f}(z)$.

Example 3.1 Let $p(z) = a_0 + a_1 z$ and $q(z) = e_0 + b_1 z$. From (3.6) we have, with $L = M = 1$;

$$(a_0 + a_1 z) - (\mathbf{c}_0 + z\mathbf{c}_1 + z^2\mathbf{c}_2 + \ldots)(e_0 + b_1 z) = O(z^3).$$

Solving the resulting system of equations yields

$$q(z) = e_0 - z\mathbf{c}_1^{-1}\mathbf{c}_2 \quad \text{and} \quad p(z) = \mathbf{c}_0 + z(\mathbf{c}_1 - \mathbf{c}_0\mathbf{c}_1^{-1}\mathbf{c}_2). \qquad []$$

For the general case, we may attempt to solve the system of equations obtained from (3.6), which contains non-commuting elements, by a method of successive elimination using designants, whose properties were studied by Heyting in (1927). Recently Salam (1993) has applied this approach to VPA's in the context of real Clifford algebras.

However, two considerations immediately present themselves, one of which concerns the invertibility of the denominator polynomial, $q(z)$. We also need to demonstrate the vectorial nature of the approximants themselves. In order to address these points we consider, instead, the construction of VPA's using vector continued fractions of the following type:

$$\mathbf{b}_0 + z\Big[\mathbf{b}_1 + z[\mathbf{b}_2 + \ldots]^{-1}\Big]^{-1}. \tag{3.7}$$

The vectors $\mathbf{b}_i \in \mathbb{C}^n$, $i = 0, 1, \ldots$ are chosen such that the successive convergents of (3.7) are the entries in the staircase sequence $[0/0]$, $[1/0]$, $[1/1]$, $[2/1]$, $\ldots$ of the VPA table. These entries may be determined using vector analogues of scalar methods — for example, the Euclidean algorithm of Graves-Morris and Roberts (1994) may be adapted to this end. In this paper we do not consider problems of degeneracy, and furthermore, for purposes of exposition, we shall only consider continued fractions of the type shown in (3.7).

The numerators and denominators of the successive convergents of (3.7) satisfy the forward recurrence relations, for $k = 1, 2, \ldots$:

$$\begin{cases} p_k(z) \;=\; p_{k-1}(z)\mathbf{b}_k + zp_{k-2}(z),\; p_{-1} := e_0,\; p_0(z) := \mathbf{b}_0 \\ q_k(z) \;=\; q_{k-1}(z)\mathbf{b}_k + zq_{k-2}(z),\;\; q_{-1} := 0,\;\; q_0(z) := e_0. \end{cases} \tag{3.8}$$

For $k = 2L$ the VPA $[L/L]$ is generated, while for $k = 2L+1$ the VPA $[L+1/L]$ is obtained.

Alternatively, we may employ the backward recurrence relations:

$$\begin{cases} X_l(z) \;=\; \mathbf{b}_{k-l}X_{l-1}(z) + zY_{l-1}(z) \\ Y_l(z) \;=\; X_{l-1}(z) \end{cases} \tag{3.9}$$

with $X_0(z) := \mathbf{b}_k$, $Y_0(z) = e_0$, and $l = 1, 2, \ldots k$. This construction yields $p_k(z) = X_k(z)$ and $q_k(z) = Y_k(z)$. The application of the above methods to form any entry in the Vector Padé Table is derived from the scalar theory and is given by Roberts (1990). These relations are used to deduce the following results for the general VPA, $p(z)[q(z)]^{-1}$, which satisfies the Baker condition:

$$\begin{cases} p(z)\tilde{p}(z) \;=\; \tilde{p}(z)p(z) \;=\; Q'(z)e_0 \;\in\; \mathbb{C} \\ q(z)\tilde{q}(z) \;=\; \tilde{q}(z)q(z) \;=\; Q(z)e_0 \;\in\; \mathbb{C} \end{cases} \tag{3.10}$$

$$p(z)\tilde{q}(z) = \sum_{i=1}^{n} P_i(z) e_i \ \in \ \mathbb{C}^n \tag{3.11}$$

in which $Q(z)$ $[Q'(z)]$ is a complex-valued polynomial of even degree $2M$ $[2L]$, while each component of $\mathbf{P}(z)$ is a complex-valued polynomial of maximum degree $(L+M)$.

Hence, unless $Q(z)$ vanishes identically, we obtain

$$[q(z)]^{-1} = \frac{\tilde{q}(z)}{Q(z)}$$

so that

$$[L/M](z) \ = \ p(z)\tilde{q}(z)/Q(z) \ = \ \mathbf{P}(z)/Q(z). \tag{3.12}$$

Thus, the invertibility of the denominator is normally assured, as is the vector nature of VPA's in general. (This latter point is already clear from the nature of (3.7).) In addition, we note that

$$\mathbf{P}(z) \cdot \mathbf{P}(z) e_0 \ = \ [p(z)\tilde{q}(z)]\,[q(z)\tilde{p}(z)] \ = \ Q(z)Q'(z)e_0.$$

Hence, we have the division property

$$Q(z) \,|\, \mathbf{P}(z) \cdot \mathbf{P}(z) \tag{3.13}$$

which is identical to (2.6) for $\mathbf{c}_i \in \mathbb{R}^n$.

Furthermore, it has been demonstrated by Roberts (1993) that both the numerator and denominator of any VPA belong to the Lipschitz group (formerly called the Clifford group) for $\mathbf{c}_i \in \mathbb{R}^n$ and to a complexified form of this group for $\mathbf{c}_i \in \mathbb{C}^n$. It then follows that $p(z)$ and $q(z)$ may each be written as a product of vectors in $\mathbb{C}^n$. As a consequence of this it may be shown that, for $\mathbf{c}_i \in \mathbb{R}^n$, $x \in \mathbb{R}$, the denominator $Q(x)$ is non-negative.

These points may be illustrated by our earlier example. For,

$$q(z) = \mathbf{c}_1^{-1}(\mathbf{c}_1 - z\mathbf{c}_2)$$

and

$$p(z) = \mathbf{c}_0\mathbf{c}_1^{-1}(\mathbf{c}_1 + z\mathbf{c}_1\mathbf{c}_0^{-1}\mathbf{c}_1 - z\mathbf{c}_2)$$

i.e., each polynomial over $C\ell(\mathbb{C}^n)$ is a product of vectors in $\mathbb{C}^n$ — use is made here of (3.4). We then obtain, up to a common factor of $\mathbf{c}_1 \cdot \mathbf{c}_1$,

$$Q(z) = (\mathbf{c}_1 \cdot \mathbf{c}_1) - 2z(\mathbf{c}_1 \cdot \mathbf{c}_2) + z^2(\mathbf{c}_2 \cdot \mathbf{c}_2)$$

and

$$\begin{aligned}\mathbf{P}(z) \ = \ & \mathbf{c}_0(\mathbf{c}_1 \cdot \mathbf{c}_1) + z\,\{\mathbf{c}_1(\mathbf{c}_1 \cdot \mathbf{c}_1) - 2\mathbf{c}_0(\mathbf{c}_1 \cdot \mathbf{c}_2)\} \\ & + \ z^2\,\{\mathbf{c}_0(\mathbf{c}_2 \cdot \mathbf{c}_2) + \mathbf{c}_2(\mathbf{c}_1 \cdot \mathbf{c}_1) - 2\mathbf{c}_1(\mathbf{c}_1 \cdot \mathbf{c}_2)\}\,.\end{aligned}$$

The non-negative nature of $Q(x)$ for $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^n$ then follows. By calculating the scalar product $\mathbf{P}(z) \cdot \mathbf{P}(z)$, the statement (3.13) is readily verified. We make the observation that, if $\mathbf{c}_i \in \mathbb{R}^n$, $i = 0, 1, \ldots$ then Vector Padé Approximants and Generalised Inverse Padé Approximants are identical (Roberts, 1990):

$$[L/M](z) \equiv \{L + M/2M\}(z).$$

The above recurrence relations which involve elements of a Clifford algebra, may be used to establish other relations determining a sequence of polynomial forms $(Q_k(z), \mathbf{P}_k(z))$, which do not involve such elements.

We set $\mathbf{P}_k(z) := p_k(z)\tilde{q}_k(z)$ and $Q_k(z) := q_k(z)\tilde{q}_k(z)$ and

$$\underline{\mathbf{P}}_k(z) := [Q_k(z), \mathbf{P}_k(z)]\,.$$

Then, following Roberts (1992), we obtain by multiplying (3.8) from the right by $\tilde{q}_k(z)$

$$\begin{cases} \underline{\mathbf{P}}_k(z) &= (\mathbf{b}_k \cdot \mathbf{b}_k)\underline{\mathbf{P}}_{k-1}(z) + z^2\underline{\mathbf{P}}_{k-2}(z) + zB^{(k-1)}(z)\mathbf{b}_k \\ B^{(k)}(z) &= 2\underline{\mathbf{P}}_{k-1}(z)\mathbf{b}_k^T + zB^{(k-1)}(z) \end{cases} \tag{3.14}$$

where each $\mathbf{b}_k$ is regarded as a column matrix and

$$\underline{\mathbf{P}}_{-1}(z) := \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix},\ \underline{\mathbf{P}}_0(z) := \begin{bmatrix} 1 \\ \mathbf{b}_0 \end{bmatrix},\ B_0(z) := \begin{bmatrix} \mathbf{0}^T \\ I_n \end{bmatrix}.$$

The backward recurrence relations similarly yield the following

$$\begin{cases} \mathbf{V}_l(z) &= \mathbf{b}_{k-l}S_{l-1}(z) + z\mathbf{V}_{l-1}(z) \\ S_l(z) &= (\mathbf{b}_{k-l} \cdot \mathbf{b}_{k-l})S_{l-1}(z) + z^2S_{l-2}(z) + 2z\mathbf{b}_{k-l} \cdot \mathbf{V}_{l-1}(z) \end{cases} \tag{3.15}$$

for $l = 1, 2, \ldots, k$, with the initialisations

$$S_{-1}(z) := 1, \quad S_0(z) := \mathbf{b}_k \cdot \mathbf{b}_k, \quad \mathbf{V}_0(z) = \mathbf{b}_k.$$

We may identify $(Q_k(z), \mathbf{P}_k(z))$ with $(S_{k-1}(z), \mathbf{V}_k(z))$.

The above discussion treats the calculation of the numerator and denominator polynomials. In particular (3.14) and (3.15) provide practical methods for their construction. We now return to the vector ϵ-algorithm and state how it may be used to construct VPA's directly. We initialise the first two columns of the vector ϵ-table (which now consists of vector-valued functions) as follows:

$$\epsilon_{-1}^{(j+1)}(z) := \mathbf{0}$$

and

$$\epsilon_0^{(j)} := \sum_{i=0}^{j} \mathbf{c}_i z^i, \quad j = 0, 1, 2, \ldots \tag{3.16}$$

The rhombus rule (1.2) is then applied using (3.2) as the definition of a vector inverse. It may then be shown (Corollary 4.5) that the even columns of the vector ϵ-table are Vector Padé Approximants; in fact

$$\epsilon_{2k}^{(j)}(z) \equiv [k + j/k](z) \tag{3.17}$$

with $k = 0, 1, \ldots$ and $j = -k, 1-k, \ldots$.

The application to the acceleration of vector sequences considered in the first section is regained by setting $z = 1$. Hence, one may see that a particular, accelerated, estimate of the limit of a sequence is, in fact, the value (at $z = 1$) of a particular VPA to the series for the generating function $\mathbf{f}(z)$ (2.2). Therefore, the convergence properties of the even columns of the ϵ-table are intimately bound up with the convergence properties of the rows of the Vector Padé Table.

3.3. A Convergence Theorem

The following theorem concerns vector-valued functions of the form

$$\mathbf{f}(z) := \frac{\mathbf{g}(z)}{R(z)} \tag{3.18}$$

where $R(z) = \prod_{i=1}^{M}(z - \alpha_i)$ in which the complex numbers α_i, $i = 1, \ldots, M$, need not be distinct but satisfy

$$0 < |\alpha_i| < \rho.$$

The $g_j(z)$, $j = 1, 2, \ldots, n$, are complex-valued functions analytic in

$$D_\rho := \{z : |z| < \rho\}$$

such that

$$\mathbf{g}(\alpha_i) \cdot \mathbf{g}(\alpha_i) \neq 0, \quad i = 1, 2, \ldots, M. \tag{3.19}$$

The Maclaurin series for $\mathbf{f}(z)$ may be determined and is of the form (2.2). We shall require the following subsets of the complex plane. We define

$$D_\rho^- := D_\rho \setminus \cup_{i=1}^{n} \{\alpha_i\}.$$

Let S be any compact subset of $D_\rho^- \cap D_\mu$ for $0 < \mu < \rho$.

Theorem 3.2 (Roberts, 1994)

(i) For $\mathbf{f}(z)$ defined by (3.18) — (3.19) the Vector Padé Approximant $[L/M](z)$ exists, for sufficiently large L, and

$$\lim_{L\to\infty} [L/M](z) = \mathbf{f}(z)$$

the convergence being uniform in compact subsets of D_ρ^-.

(ii) The rate of convergence is governed by

$$\lim_{L\to\infty} \sup_{z\in S} |\mathbf{f}(z) - [L/M](z)|^{\frac{1}{L}} \leq \frac{\mu}{\rho}. \tag{3.20}$$

(iii) Furthermore, if the denominator polynomial, denoted here by $q^{[L/M]}(z)$, is monic then

$$\lim_{L\to\infty} q^{[L/M]}(z) = R(z)e_0$$

the convergence being uniform in any bounded subset E of the complex plane; the rate of convergence is governed by

$$\lim_{L\to\infty}\sup_{z\in E}|R(z)e_0 - q^{[L/M]}(z)|^{\frac{1}{L}} \le \max_{1\le i\le M}\frac{|\alpha_i|}{\rho}. \qquad \square$$

In the above we use the absolute or spinor norm on $C\ell(\mathbb{C}^n)$ which is identical to the Euclidean norm for the case of vectors (Hile and Lounesto, 1990). There are corresponding results for the behaviour of the numerator polynomial as well as of the appropriate polynomial form $(Q(z), \mathbf{P}(z))$.

The proof of this theorem, which uses a matrix representation of $C\ell(\mathbb{C}^n)$, is based on the proof by Saff (1972) of de Montessus' theorem for the scalar case and is simpler than the proofs by Graves-Morris and Saff (1988, 1991) of the corresponding results using the generalised inverse (2.1). This realises the hope expressed in the introduction.

However, the vector-valued generating function, (2.2), corresponding to a sequence $\{\mathbf{s}_i\}$ satisfying the recurrence relations (2.4) has been shown by Graves-Morris (1983) to be of the form

$$\mathbf{f}(\mathbf{z}) = \frac{\mathbf{p}(\mathbf{z})}{\mathbf{q}(\mathbf{z})} \tag{3.21}$$

where each $p_i(z)$, $i = 1, \ldots n$ and $q(z)$ are complex-valued polynomials of maximum degree K. Hence, the $[L/K]$ VPA to the generating function series yields $\mathbf{f}(z)$ exactly for $L \ge K$. Now, if the limit of the sequence $\{\mathbf{s}_i\}$ is $\mathbf{s}_\infty$, then it may be shown that

$$\mathbf{s}_\infty = \mathbf{f}(1) = \mathbf{a}.$$

It therefore follows that, provided no zero divisions are encountered in the construction of the ϵ-table,

$$\epsilon_{2K}^{(j)} = \mathbf{a} \quad \text{for} \quad j = 0, 1, \ldots$$

thus proving McLeod's result (2.5). We refer the reader to Graves-Morris (1983) and Graves-Morris and Jenkins (1986) for a full discussion of this topic.

We now consider a sequence generated by

$$\mathbf{s}_{i+1} := E\mathbf{s}_i + \mathbf{e}, \quad i = 0, 1, 2, \ldots \tag{3.22}$$

where $E \in \mathbb{C}^n \times \mathbb{C}^n$ and $\mathbf{e} \in \mathbb{C}^n$. This is the type of sequence which occurs in an iterative solution to a system of linear equations. If the matrix $(I - E)$ is non-singular then (3.22) has a unique fixed point $\mathbf{s}_\infty$ satisfying

$$\mathbf{s}_\infty = E\,\mathbf{s}_\infty + \mathbf{e}.$$

In the manner of Graves-Morris and Jenkins (1986) we assume that E has non-null eigenvalues, λ_i, satisfying

$$\begin{aligned} &|\lambda_1| \ge |\lambda_2| \ge \ldots \ge |\lambda_{J_0}| > 1 > |\lambda_{J_0+1}| = |\lambda_{J_0+2}| \ldots \\ &= |\lambda_{J_1}| > |\lambda_{J_1+1}| = \ldots = |\lambda_{J_2}| > \ldots |\lambda_{J_l}| \end{aligned} \tag{3.23}$$

where $K = J_l$.

The sequence $\{\mathbf{s}_i\}$ does satisfy the recurrence relations (2.4) as shown by Smith *et al.* (1987), and hence McLeod's theorem is applicable. However, for a practical system, K may be very large. We may apply theorem 3.2 to obtain results on the convergence behaviour of even columns of the vector ϵ-table. This is possible since $\mathbf{f}(z)$ has the form (3.21) — we assume that we are fortunate enough for (3.19) not to be violated. As demonstrated by Graves-Morris (1992) the zeros of the denominator are $\alpha_i = \lambda_i^{-1}$ for $i = 1, 2, \ldots, K$. We now consider a row sequence of VPA's to $\mathbf{f}(z)$ in the disks $|z| < |\alpha_i|$, $i = J_1, J_2, \ldots, J_l = K$, each evaluated at $z = 1$. Then, those even columns of the ϵ-table whose indices are not less than $2J_0$ are convergent. From the above theorem we obtain

$$\lim_{j\to\infty} \sup_{z\in S} |\mathbf{f}(z) - [j + J_i/J_i](z)|^{\frac{1}{j}} \leq \mu|\lambda_{J_{i+1}}|$$

for $i = 0, 1, 2, \ldots, l-1$, with $1 < \mu < |\lambda_{J_1}|^{-1}$. Since we may chose μ as close to unity as we please it follows that the rate of convergence of the even columns is governed by:

$$\lim_{j\to\infty} |\mathbf{f}(1) - \epsilon_{2J_i}^{(j)}|^{\frac{1}{j}} \leq |\lambda_{J_{i+1}}|$$

for $i = 0, 1, \ldots, l-1$.

Numerical examples illustrating these points are given by Graves-Morris (1992).

4. Vector Rational Hermite Interpolants

In this section we consider a generalisation of the vector ϵ-algorithm to Hermite interpolation.

4.1. Definitions

Suppose we are given the sequence $Z := \{z_0, z_1, z_2, \ldots\}$, where the z_i are complex numbers (not necessarily distinct), and the formal vector-valued Newton series

$$\mathbf{f}(z) := \sum_{i=0}^{\infty} \mathbf{c}_{0,i} B_i(z) \tag{4.1}$$

where the $\mathbf{c}_{0,i} \in \mathbb{C}^n$ are vector divided differences, and the polynomials $B_i(z)$ are given by

$$B_0(z) = 1, \quad B_i(z) = \prod_{j=0}^{i-1} (z - z_j), \quad i = 1, 2, \ldots . \tag{4.2}$$

Then we define the $[L/M]$ *vector rational Hermite interpolant* to $\mathbf{f}(z)$ as that vector-valued rational function $[L/M](z) := p(z)[q(z)]^{-1}$ where $p(z)$ and $q(z)$ are polynomials in z, taken over $C\ell(\mathbb{C}^n)$, of degrees $\partial p \leq L$ and $\partial q \leq M$, such that

$$\frac{d^{k_i}}{dz^{k_i}}[L/M](z_i) = \frac{d^{k_i}}{dz^{k_i}}\mathbf{f}(z_i) \tag{4.3}$$

for $k_i = 0, 1, \ldots (m_i - 1)$ and $i = 0, 1, 2, \ldots, (L+M)$, in which the multiplicity of z_i in $\{z_0, \ldots z_{L+M}\}$ is m_i.

By considering the Newton series for $[L/M](z)$ we see that (4.3) is equivalent to

$$[L/M](z) - \mathbf{f}(z) = 0(B_{L+M+1}(z)) \tag{4.4}$$

where $0(B(z))$ signifies a function which vanishes at the zeros of $B(z)$ with corresponding multiplicity. As in the scalar case, if $[L/M](z)$ satisfies (4.4) then $p(z), q(z)$ satisfy the *linearised* (or *modified*) *vector Hermite problem* viz.

$$p(z) - \mathbf{f}(z)q(z) = 0(B_{L+M+1}(z)) \tag{4.5}$$

see e.g., Gutnecht (1989) and Warner (1974). However, a solution to (4.5) does not necessarily solve the vector rational Hermite problem.

Examples of the cases of full confluence (Padé) and of Cauchy interpolation (all z_i distinct) are given by Roberts (1990), while illustrations of the use of reciprocal differences in the construction of $[L/M](z)$ are given by the same author in 1992. These examples involve continued fractions which may normally be used, as in the scalar case, to construct Hermite interpolants, if they exist, for any value of L and M. The vector continued fractions are of the form

$$\mathbf{b}_0 + (z - z_0)\left[\mathbf{b}_1 + (z - z_1)[\mathbf{b}_2 + \ldots]^{-1}\right]^{-1}$$

in which the $\mathbf{b}_i$, $i \geq 0$, may be constructed from the interpolatory data using inverse or reciprocal differences. The numerator and denominator polynomials enjoy the same properties as those for Vector Padé Approximants — viz. the existence of forward and backward recurrence relations (3.8, 3.9) and their practical versions (3.14, 3.15), including the division property (3.13). In addition, as shown by Roberts (1993) these polynomials belong to the extended form of the Lipschitz group.

Theorem 4.1 If it exists, the solution to the vector rational Hermite problem is unique.

Proof For a proof of this theorem, see Roberts (1993). []

As in the scalar case, the $[L/M](z)$ form a two-dimensional array for L, $M \geq 0$ — c.f. Fig.3. We shall only consider *normal* tables — i.e. those for which every entry $[L/M]$ exists with $p(z)$, $q(z)$ each of full maximum degree, together with the requirement that their leading coefficients are invertible and not merely non-zero.

4.2. The Vector Claessens' Identity

We assume that the formal Newton series (4.1) for a vector-valued function is normal and consider the five entries denoted by compass points as shown in Fig.4.

Theorem 4.2 [Claessens' identity] (Graves-Morris and Jenkins, 1986) (Claessens, 1978)

$$\begin{aligned}(z - z_{L+M+1})^{-1}\left\{(E - C)^{-1} - (S - C)^{-1}\right\} &= \\ (z - z_{L+M})^{-1}\left\{(N - C)^{-1} - (W - C)^{-1}\right\}&\end{aligned} \tag{4.6}$$

$$\begin{array}{ccccccccc} & [L/M-1](z) & & & & & N & \\ [L-1/M](z) & [L/M](z) & [L+1/M](z) & & \leftrightarrow & W & C & E \\ & [L/M+1](z) & & & & & S & \end{array}$$

Fig. 4.

Proof The proof follows the method used in Graves-Morris and Roberts (1994) to establish Cordellier's identity for full confluence, using the generalised inverse. From (4.5) and using the compass notation, we have

$$\tilde{p}_N(z)q_C(z) - \tilde{q}_N(z)p_C(z) \;=\; 0(B_{L+M}(z)) \tag{4.7}$$

However, the left-hand side is a polynomial of maximum degree $(L+M)$, whose coefficients lie in $C\ell(\mathbb{C}^n)$. Comparing leading coefficients we obtain

$$\tilde{p}_N(z)q_C(z) - \tilde{q}_N(z)p_C(z) \;=\; \dot{\tilde{p}}_N\dot{q}_C B_{L+M}(z) \tag{4.8}$$

in which $\dot{p}$ denotes the leading coefficient of $p(z)$.

Similarly,

$$\tilde{p}_C(z)q_W(z) - \tilde{q}_C(z)p_W(z) \;=\; \dot{\tilde{p}}_C\dot{q}_W B_{L+M}(z) \tag{4.9}$$

$$\tilde{p}_N(z)q_W(z) - \tilde{q}_N(z)p_W(z) \;=\; \dot{\tilde{p}}_N\dot{q}_W B_{L+M}(z) \tag{4.10}$$

We may restate the right-hand side of (4.6) as

$$(z - z_{L+M})^{-1}\{(N-C)^{-1}(W-N)(W-C)^{-1}\} \tag{4.11}$$

which, on using (4.8), (4.9), and (4.10) becomes

$$\begin{aligned} &(z - z_{L+M})^{-1} \cdot \left[\tilde{q}_N^{-1}\tilde{p}_N - p_C q_C^{-1}\right]^{-1}\left[p_W q_W^{-1} - \tilde{q}_N^{-1}\tilde{p}_N\right]\left[p_W q_W^{-1} - \tilde{q}_C^{-1}\tilde{p}_C\right]^{-1} \\ &= (z - z_{L+M})^{-1} q_C \left[\tilde{p}_N q_C - \tilde{q}_N p_C\right]^{-1}\left[\tilde{q}_N p_W - \tilde{p}_N q_W\right]\left[\tilde{q}_C p_W - \tilde{p}_C q_W\right]^{-1}\tilde{q}_C \\ &= \{B_{L+M+1}(z)\}^{-1} \cdot q_C(z)\dot{q}_C^{-1}\dot{\tilde{p}}_C^{-1}\tilde{q}_C(z). \end{aligned} \tag{4.12}$$

Again, using (4.5), we obtain

$$\tilde{p}_E(z)q_C(z) - \tilde{q}_E(z)p_C(z) \;=\; \dot{\tilde{p}}_E\dot{q}_C B_{L+M+1}(z) \tag{4.13}$$

$$\tilde{q}_C(z)p_S(z) - \tilde{p}_C(z)q_S(z) \;=\; -\dot{\tilde{p}}_C\dot{q}_S B_{L+M+1}(z) \tag{4.14}$$

$$\tilde{q}_E(z)p_S(z) - \tilde{p}_E(z)q_S(z) \;=\; -\dot{\tilde{p}}_E\dot{q}_S B_{L+M+2}(z). \tag{4.15}$$

Using an argument similar to that above, it follows that:

$$\begin{aligned} &(z - z_{L+M+1})^{-1}\{(E-C)^{-1} - (S-C)^{-1}\} \;= \\ &\qquad \{B_{L+M+1}(z)\}^{-1} \cdot q_C(z)\dot{q}_C^{-1}\dot{\tilde{p}}_C^{-1}\tilde{q}_C(z). \end{aligned} \tag{4.16}$$

The identity (4.6) follows from (4.12) and (4.16). $\square$

We immediately deduce Wynn's identity for Vector Padé Approximants by locating all z_i at the origin.

Corollary 4.3

$$[E-C]^{-1}-[S-C]^{-1} = [N-C]^{-1}-[W-C]^{-1}$$

Claessens' identity (4.6) may be used to construct vector rational Hermite interpolants — including Vector Padé Approximants in the case of full confluence — by employing the initialisation:

$$[L/0](z) := \sum_{i=0}^{L} \mathbf{c}_{0,i} B_i(z), \quad L = 0,\, 1,\, \ldots$$

and the artificial boundary conditions:

$$[L/-1](z) := \mathbf{0} \quad \text{and} \quad [-1/M] := \infty, \quad L, M = 0,\, 1,\, \ldots\,.$$

An equivalent approach is to implement the generalised vector ϵ-algorithm, which we now describe.

4.3. Generalised Vector ϵ-algorithm

We again consider the formal Newton series (4.1) for a vector-valued function, which has a normal table. Then, a two-dimensional array — an extended form of Fig.2., of vector entities $\epsilon_k^{(j)}(z) \in \mathbb{C}^n$, may be constructed, as in Fig.5., using the algorithm

$$\epsilon_{k+1}^{(j)}(z) := \epsilon_{k-1}^{(j+1)}(z) + (z - z_{k+j+1})^{-1}\left[\epsilon_k^{(j+1)}(z) - \epsilon_k^{(j)}(z)\right]^{-1} \tag{4.17}$$

for $k = 0,\, 1,\, \ldots\; j = -k, 1-k, 2-k, \ldots$ with the initialisations

$$\begin{cases} \epsilon_{-1}^{(j)}(z) := \mathbf{0}, & \epsilon_{2j}^{(-j-1)}(z) := \mathbf{0} \\ \epsilon_0^{(j)}(z) := \sum_{i=0}^{j} \mathbf{c}_{0,i} B_i(z) \ \text{ for } & j = 0, 1, 2, \ldots\,. \end{cases} \tag{4.18}$$

The next theorem identifies entries in the even columns of the vector ϵ-table with vector rational Hermite interpolants.

Theorem 4.4

$$\epsilon_{2k}^{(j)}(z) = [j+k/k](z) \tag{4.19}$$

for $k = 0,\, 1,\, 2,\, \ldots$ and $j = -k, 1-k, 2-k, \ldots\,.$

Proof We follow the method of proof given in volume 13 of Baker and Graves-Morris (1981) for scalar Padé Approximants. For $k = 0$, (4.19) is clearly valid — $\epsilon_0^{(j)}(z)$ is a vector-valued Hermite polynomial by definition. Consider $\epsilon_2^{(j)}(z)$, which is constructed as follows, using (4.17) and (4.18):

$$\epsilon_1^{(j)}(z) = [B_{j+2}(z)\mathbf{c}_{0,j+1}]^{-1}, \quad j = -1,\, 0,\, 1,\, \ldots$$

$$
\begin{array}{ccccccccc}
 & \epsilon_0^{(-1)} & & \epsilon_2^{(-2)} & & \epsilon_4^{(-3)} & \cdot \\
\epsilon_{-1}^{(0)} & & \epsilon_1^{(-1)} & & \epsilon_3^{(-2)} & & & \cdot \\
 & \epsilon_0^{(0)} & & \epsilon_2^{(-1)} & & \epsilon_4^{(-2)} & \cdot \\
\epsilon_{-1}^{(1)} & & \epsilon_1^{(0)} & & \epsilon_3^{(-1)} & & & \cdot \\
 & \epsilon_0^{(1)} & & \epsilon_2^{(0)} & & \cdot & \cdot \\
\cdot & & \cdot & & \cdot & & & \cdot
\end{array}
$$

Fig. 5. The Full Vector ϵ-table

and so

$$
\begin{aligned}
\epsilon_2^{(j)}(z) &= \epsilon_0^{(j+1)}(z) + (z-z_{j+2})^{-1}[\epsilon_1^{(j+1)}(z) - \epsilon_1^{(j)}(z)]^{-1} \\
&= \sum_{i=0}^{j+1} \mathbf{c}_{0,i}B_i(z) + B_{j+2}(z)\mathbf{c}_{0,j+2}[e_0 - (z-z_{j+2})(\mathbf{c}_{0,j+1})^{-1}\mathbf{c}_{0,j+2}]^{-1}.
\end{aligned} \tag{4.20}
$$

If we expand $[\,]^{-1}$ using the appropriate form of the binomial theorem, we obtain

$$
\epsilon_2^{(j)}(z) - \mathbf{f}(z) = B_{j+3}(z)\left(\left[\mathbf{c}_{0,j+2}(\mathbf{c}_{0,j+1})^{-1}\mathbf{c}_{0,j+2} - \mathbf{c}_{0,j+3}\right] + 0(z-z_{j+2})\right) + 0(B_{j+4}(z))
$$

i.e., $\epsilon_2^{(j)}(z)$ satisfies the order condition (4.4) with $L = j+1$ and $M = 1$.

Now, it is clear that $\epsilon_2^{(j)}(z)$, given by (4.20), is a vector-valued rational function whose denominator is linear in z. The numerator polynomial is given by:

$$
\sum_{i=0}^{j+1} \mathbf{c}_{0,i}B_i(z)\left[e_0 - (z-z_{j+2})(\mathbf{c}_{0,j+1})^{-1}\mathbf{c}_{0,j+2}\right] + B_{j+2}(z)\mathbf{c}_{0,j+2}.
$$

The coefficient of the highest power, z^{j+2}, is

$$
\mathbf{c}_{0,j+1}\left[-(\mathbf{c}_{0,j+1})^{-1}\mathbf{c}_{0,j+2}\right] + \mathbf{c}_{0,j+2} = \mathbf{0}.
$$

Hence,

$$
\epsilon_2^{(j)}(z) \equiv [j+1/1](z) \quad \text{for} \quad j = -1, 0, 1, \ldots .
$$

We have now shown that (4.19) is valid for $k = 0, 1$. Induction is used to establish its validity for $k > 1$. This involves the repeated use of (4.17), which may be shown to be consistent with Claessens' identity (4.6). Thus the ϵ-algorithm (4.17) may be used to calculate columns of the ϵ-table working from left to right, while Claessens' identity (4.6) enables entries of the vector rational Hermite interpolant table to be computed working from the top two rows downwards. Hence, if (4.19) is assumed valid for $k \le K$ and $j = -k, 1-k, \ldots$ then, by direct construction of the elements of the next even column of the ϵ-table (whenever this is possible) we see that the relation must be true for $k = K+1$. Therefore, (4.19) holds for $k \ge 0$. []

For the case of full confluence, i.e., $z_i = 0$, $i \ge 0$, we obtain:

Corollary 4.5 If we construct a vector ϵ-table from the series (2.2), using

$$
\epsilon_{k+1}^{(j)}(z) := \epsilon_{k-1}^{(j+1)}(z) + \left[\epsilon_k^{(j+1)}(z) - \epsilon_k^{(j)}(z)\right]^{-1}, \tag{4.21}
$$

where $k = 0, 1, \ldots, j = -k, 1-k, \ldots,$ with the initialisations:

$$\epsilon_{-1}^{(j)}(z) := \mathbf{0}, \quad \epsilon_{2j}^{(-j-1)}(z) := \mathbf{0}, \quad \epsilon_0^{(j)} := \sum_{i=0}^{j} \mathbf{c}_i z^i \tag{4.22}$$

then

$$\epsilon_{2k}^{(j)}(z) \equiv [j+k/k](z). \tag{4.23}$$

We have proved an algorithm, (4.17), which may be used to calculate vector rational Hermite functions — involving either numerical or functional evaluation. This algorithm does not involve Clifford algebra elements — for the inverse of a vector we use (3.2) — but its validity for a normal table was established using such an algebra — c.f., Roberts (1992).

5. Variations on a Theme

There are representations of vectors other than that portrayed in the previous two sections. For example, Roberts (1990) uses $e_0, e_1, \ldots e_{n-1}$ with $e_i^2 = -e_0, i = 1, \ldots, n-1$ to describe vectors $\mathbf{v} \in \mathbb{C}^n$. However, in this section, we shall concentrate on various contributions from the use of Clifford algebras to generalised inverse rational forms, for which the vector inverse is defined by (2.1). First of all, we describe the McLeod isomorphism for $\mathbf{v} \in \mathbb{C}^n$.

Let $d = 2n+1$ and suppose $\mathbf{v} = \mathbf{x} + i\mathbf{y}\,\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then, the Clifford element corresponding to $\mathbf{v}$ is given by

$$V = \sum_{j=1}^{n} (x_j e_j + y_j e_d e_{n+j}).$$

where V belongs to the real Clifford algebra $C\ell_d$ of $\mathbb{R}^d$. We then obtain

$$\tilde{V} \leftrightarrow \mathbf{v}^* \quad \text{and} \quad V\tilde{V} = \mathbf{v} \cdot \mathbf{v}^* e_0$$

and hence

$$V^{-1} \leftrightarrow \mathbf{v}^{-1}.$$

McLeod (1971) adopted this isomorphism to prove Theorem 2.1 for real β_i. This approach is also used by Graves-Morris and Roberts (1994) to prove the block structure for generalised inverse Padé Approximants and to deduce the validity of Cordellier's identity for these blocks. This is achieved using analogues of the Kroenecker and Euclid algorithms, which are shown to be reliable algorithms for the construction of these approximants. These results also hold for the VPA's of section 3 when the coefficients $\mathbf{c}_i$ of the power series (2.2) are real vectors.

Generalised inverse approximants may also be constructed using $C\ell_d$ with $d = 2n$ as follows:

$$V \leftrightarrow \mathbf{v}$$

where

$$V := \sum_{j=1}^{n} (x_j e_j + y_j e_{n+j}).$$

We obtain

$$V^2 = \mathbf{v}.\mathbf{v}^* e_0 \quad \text{and} \quad V^{-1} \leftrightarrow \mathbf{v}^{-1}$$

where

$$\mathbf{v}^{-1} := \frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}^*}$$

which is not the same as (2.1). Although we do not represent the Moore-Penrose inverse exactly, the resulting polynomial forms $(\mathcal{Q}(z), \boldsymbol{\mathcal{P}}(z))$, where $\boldsymbol{\mathcal{P}}(z) \in \mathbb{R}^{2n}$, may be identified with the generalised inverse version $(Q(z), \mathbf{P}(z))$ as follows:

$$Q(z) := \mathcal{Q}(z), \quad P_j(z) := \mathcal{P}_j(z) + i\mathcal{P}_{n+j}(z), \quad j = 1, \ldots n.$$

The advantage of this association is that we may now employ the recurrence relations (3.14, 3.15), as well as the generalised vector ϵ-algorithm, to calculate not only approximants but also interpolants of the generalised inverse variety. However, we then require that the interpolant points are symmetric about the real axis — see Roberts (1992). Hence, if we wish to add one point at a time to the set of interpolation points being used, we are restricted to real points — this view coincides with the results of Graves-Morris and Jenkins (1986).

Finally, we comment on the possibility of an algebraic foundation for generalised inverse rational interpolants similar to that of section 4 for vector rational Hermite interpolants which is valid for any set of interpolation points. A problem emerges when an isomorphism is sought which represents not only the inverse of a complex vector defined by (2.1) but also multiplication of such a vector by a complex scalar. This problem may be circumvented if multiplicative associativity is abandoned (Roberts, 1993). To be precise we build a non-associative algebra on $C\ell_n \times C\ell_n$ whose elements are denoted by (a, b) where $a, b \in C\ell_n$. Addition is performed componentwise while multiplication is defined by

$$(a, b)(c, d) := (ac - d\tilde{b}, \tilde{a}d + cb), \quad \text{for } (a, b), (c, d) \in C\ell_n \times C\ell_n.$$

The reverse anti-automorphism is defined by

$$\tilde{A} := (\tilde{a}, -b) \quad \text{if} \quad A = (a, b).$$

The vector $\mathbf{v}$ is represented by $V := (\mathbf{x}, \mathbf{y})$. We obtain

$$\tilde{V} \leftrightarrow \mathbf{v}^* \quad \text{and} \quad V\tilde{V} = \tilde{V}V = \mathbf{v} \cdot \mathbf{v}^* e_0.$$

Hence,

$$V^{-1} \leftrightarrow \mathbf{v}^{-1}$$

with $\mathbf{v}^{-1}$ given by (2.1). Also, if $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, and $\Lambda := (\alpha e_0, \beta e_0)$, then

$$\Lambda V \leftrightarrow \lambda \mathbf{v}$$

as required — thus allowing complex multiplication in (2.3).

In conclusion, the simplest form of algebraic framework for a theory of vector rational approximation, whose structure is similar to that of the scalar case is the one erected in sections 4 and 5. Although this corresponds to a vector inverse of the Moore-Penrose type for real vectors only, the drawback of the possibility of the lack of invertibility of non-null complex vectors is off-set by the convergence theorem 3.1 which guarantees that this does not happen in certain applications.

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