A STIFFENED PLATE ANALOGY FOR PLATE-AND-BEAM STRUCTURES

J. S. Kuang[1] and Johnson H. Zhang[2]

ABSTRACT

The current design practice of plate-and-grid structures in buildings, such as waffle slabs and ribbed plates, is relied largely on code-specified approximate methods, and the deflection of the system is rarely evaluated owing to the complexity of calculations. In this paper, an analogous stiffened plate is presented to model the plate-and-grid structures. A semi-analytical method is proposed based on the finite element method to obtain an accurate analysis of the stiffened plates. The proposed method of analysis has more flexible geometry adaptability and higher efficiency in reducing the unknown variables as compared to the general finite element methods, as well as less computation time and cost. The predictions of the proposed analyses for example structures agree very well with those of the finite element method and experiment.

KEY WORDS

Plate-and-beam structures, stiffened plates, finite element method.

INTRODUCTION

Plate-and-grid structures, such as ribbed plates and waffle slabs, are widely used in building systems. In current design practice, the analysis of plate-and-grid structures is relied largely on code-specified approximate methods, and the deflection of a plate-and-grid system is rarely evaluated in practice owing to the complexity of calculations (MacGregor 1997). In general, the code-specified approximate methods of analysis are most appropriate for the uniform or quasi-uniform structures. For the structures other than the regular ones, it is necessary to use a more sophisticated model to obtain a precise analysis.

This paper presents a stiffened-plate model for the analysis of plate-and-grid structures. An effective semi-analytical method is proposed based on the finite element approach for the analysis of the stiffened plates. The proposed method can be used to solve effectively the problem of the stiffened plates with arbitrary configuration, openings, material discontinuity and arbitrary orientated stiffeners by utilising the naturally existed intrinsic relationship among the nodes on the same nodal line with one analytical expression. The higher efficiency of the proposed method has also been found on reducing the unknown variables as compared to the boundary element method. By employing the analytical transformation instead of matrix inverse, the approach has shown significant saving in the computation time

[1] Department of Civil Engineering, Hong Kong University of Science & Technology, Kowloon, Hong Kong, Phone +852 2358-7162, FAX +852 2358-1534, cejkuang@ust.hk
[2] School of the Built Environment, Napier University, Edinburgh EH10 5DT, UK, Phone +44 131/455-2482, FAX +44 131/455-2239, j.zhang@napier.ac.uk

as compared to the static condensation method. The predictions of the proposed method agree very well the FEM and experimental results.

METHOD OF ANALYSIS

A general plate-and-grid structure consisting of slabs and beams is shown in Figure 1. The structure can conveniently be considered as a stiffened plate with the combination of plate panels and grids (or stiffeners). Since the skeleton of grids is considered as a naturally strip-like structure, the stiffened plate is always divided into a number of strips by nodal lines, such as lines *L* and *L'* shown in Figure 1. The openings, material discontinuities and arbitrary orientated stiffeners can then be placed anywhere within the strips, which will later be discreted into triangular or quadratic plate elements.

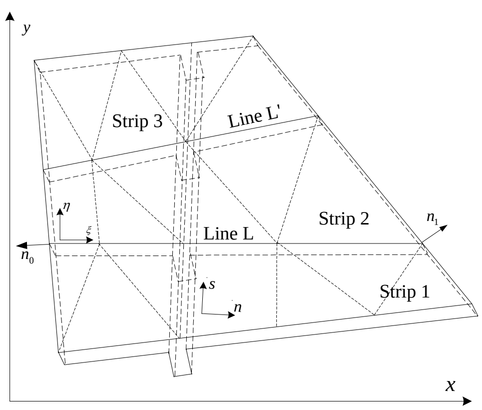


Figure 1: Analogous Stiffened Plate for General Plate-and-Grid Structures

NODAL LINE DISPLACEMENT

Consider Line *L* in Figure 1. The displacement along this line can be described by the following equations:

$$w_i = \sum_{m=1}^{r} a_m X_m(\xi_i), \quad (\frac{\partial w}{\partial \xi})_i = \sum_{m=1}^{r} a_m X'_m(\xi_i), \quad \text{and} \quad (\frac{\partial w}{\partial \eta})_i = \sum_{n=1}^{p} b_n \Phi_n(\xi_i) + \theta_0(1-\frac{\xi_i}{l}) + \theta_1\frac{\xi_i}{l} \quad (1)$$

where ξ_i is the local coordinate of the *i*th node on the nodal line; *N* is the number of nodes; *l* is the length of the nodal line; and $X_m(\xi)$ and $\Phi_n(\xi)$ are the displacement base functions of the nodal line. They must satisfy the requirements of boundary constraint at the ends of the line. The parameters a_m and b_n are used to describe directly the nodal line displacements, and rotations θ_o and θ_1 corresponding to the starting and end points, respectively are defined as

$$\begin{cases} \theta_o = \dfrac{m_o}{l_0}\displaystyle\sum_{m=1}^{r} a_m X'_m(0) \text{ (simply supported) or 0 (free or clamed)} \\ \theta_1 = \dfrac{m_1}{l_1}\displaystyle\sum_{m=1}^{r} a_m X'_m(l) \text{ (simply supported) or 0 (free or clamed)} \end{cases} \quad (2)$$

$$l_0 = \frac{|\vec{n}_0 \times \vec{\xi}|}{|\vec{n}_0| \times |\vec{\xi}|},\ m_0 = \frac{|\vec{n}_0 \times \vec{\eta}|}{|\vec{n}_0| \times |\vec{\eta}|};\ l_1 = \frac{|\vec{n}_1 \times \vec{\xi}|}{|\vec{n}_1| \times |\vec{\xi}|},\ m_1 = \frac{|\vec{n}_1 \times \vec{\eta}|}{|\vec{n}_1| \times |\vec{\eta}|} \tag{3}$$

BOUNDARY RESTRAINTS

The displacement base functions, $X_m(\xi)$ and $\Phi_n(\xi)$, are significantly influenced by the boundary restraints. At the starting point of a nodal line ($\xi = 0$), the restraints for different boundary conditions are as follows.

- When the point is clamped, $w = 0$, $\partial w / \partial \xi = 0$, and $\partial w / \partial \eta = 0$. The displacement boundary conditions are $X_m(0) = 0$, $X'_m(0) = 0$, and $\Phi_n(0) = 0$.
- When the point is simply supported, $w = 0$, $-(\partial w / \partial \xi) m_0 + (\partial w / \partial \eta) l_0 = 0$. The displacement boundary conditions are $X_m(0) = 0$, and $\Phi_n(0) = 0$.
- When the point is free, there is no any restraint to w, $\partial w / \partial \xi$ and $\partial w / \partial \eta$, thus having no boundary constraint to the base functions, $X_m(0)$, $X'_m(0)$ and $\Phi_n(0)$.

In the proposed method, it is assumed that the approximate deformed profiles of nodal lines are known. Thus, the base functions $X_m(\xi)$ and $\Phi_n(\xi)$ will govern the profiles of the deformation, while the parameters a_m and b_n will be used to determine the exact deflected shapes. The parameters a_m and b_n are then become unknown variables instead of the node displacements. Therefore, if the profile of the combination of base functions is closer to that of the nodal line displacement, the number of unknown variables can be reduced.

Theoretically, various functions may be used as the displacement base functions, provided that they can portray the profile of the deformed nodal line and satisfy the boundary conditions. In this paper, the shape functions of the beam in bending are selected as the base functions of the nodal line (Weaver 1990).

TRANSFORMATION TECHNIQUE

In order to obtain the displacement of each node in nodal lines, the unknown variables a_m and b_n can firstly be determined by solving the simultaneous equations of a_m and b_n. In this paper, a new transformation technique is proposed for the development of such simultaneous equations. In the proposed technique, two transformation steps are needed in the analysis. The transformation at the nodal line level should firstly be conducted; then the transformation at the element level is carried out with an assembly of the nodal transformations.

TRANSFORMATION OF NODE DISPLACEMENT TO NODAL LINE DISPLACEMENT

Consider a typical strip of a stiffened plate shown in Figure 2. Theoretically, two nodal lines *L* and *L'* are not necessary to parallel each other. For the sake of brevity, the two nodal lines are assumed to be parallel to the x-axis. The origin point of the local coordinate system is located at one of the ends of the nodal line *L* and the *x*-axis passes through this nodal line, as shown in Figure 2. The base function Equation (1) can be rewritten as

$$w_i = \sum_{m=1}^{r} a_m X_m(x_i),\ \left(\frac{\partial w}{\partial x}\right)_i = \sum_{m=1}^{r} a_m X'_m(x_i),\ \text{and}\ \left(\frac{\partial w}{\partial y}\right)_i = \sum_{n=1}^{p} b_n \Phi_n(x_i) + \theta_0\left(1-\frac{x_i}{l}\right)+\frac{\theta_1 x_i}{l} \quad (4)$$

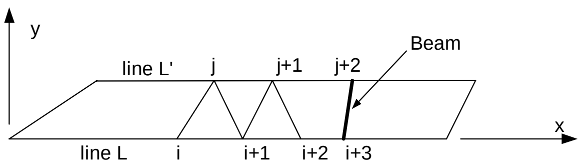


Figure 2: A typical Strip

The ith nodal displacement on the lth nodal line is $[\delta_i] = \{w_i, (\partial w/\partial x)_i, (\partial w/\partial y)_i\}$, which can be expressed in a matrix form as

$$[\delta_i] = \mathbf{T}_i \mathbf{V}^l \quad (5)$$

where

$$\mathbf{T}_i = \begin{bmatrix} X_1(x_i) & X_2(x_i) & \cdots & X_r(x_i) & 0 & 0 & \cdots & 0 \\ X'_1(x_i) & X'_2(x_i) & \cdots & X'_r(x_i) & 0 & 0 & \cdots & 0 \\ C_1(x_i) & C_2(x_i) & \cdots & C_r(x_i) & \Phi_1(x_i) & \Phi_2(x_i) & \cdots & \Phi_p(x_i) \end{bmatrix} \quad (6)$$

and the generalised nodal line displacement vector $\mathbf{V}^l$ is expressed as

$$\mathbf{V}^l = \{a_1, a_2, \cdots, a_r, b_1, b_2, \cdots, b_p\} \quad (7)$$

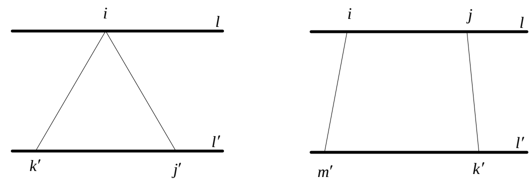
Equation (5) presents the relationship between the node displacement $[\delta_i]$ and the nodal line displacement vector $\mathbf{V}^l$. It is seen that the transformation at the element level should be performed by use of Equation (5), which is actually a process of the assembly of the transformations at the nodal line level.

TRANSFORMATION OF NODE DISPLACEMENT TO STRIP DISPLACEMENT

After deriving the relationship between the node displacement and the nodal line displacement vector, the stiffness matrix and force vector are then needed to transform accordantly. In order to transform the stiffness matrix and force vector, the relationship between the displacements of nodes and strips is derived and expressed by

$$\boldsymbol{\delta}^e = \mathbf{T}_e \mathbf{V}^s \quad (8)$$

where δ^e is the element displacement vector for plates or beams, $\mathbf{T}_e$ is the transformation matrix, and the strip displacement vector $\mathbf{V}^s = \{\mathbf{V}^l, \mathbf{V}^{l+1}\}$. In the finite element analysis, the most commonly used elements in the plate discretisation are triangular and quadrilateral ones. The typical triangular and quadrilateral elements are shown in Figure 3.



(a) Triangular (b) Quadrilateral

Figure 3: Common Elements in Plate Discretisation

The element displacement vectors and the corresponding transformation matrices of the triangular and quadrilateral elements can be expressed, respectively, by

$$\boldsymbol{\delta}^e = \{\delta_i, \delta_{j'}, \delta_{k'}\},\ \mathbf{T}_e = \begin{bmatrix} \mathbf{T}_i & 0 \\ 0 & \mathbf{T}_{j'} \\ 0 & \mathbf{T}_{k'} \end{bmatrix};\ \text{and}\ \ \boldsymbol{\delta}^e = \{\delta_i, \delta_j, \delta_{k'}, \delta_{m'}\},\ \mathbf{T}_e = \begin{bmatrix} \mathbf{T}_i & 0 \\ \mathbf{T}_j & 0 \\ 0 & \mathbf{T}_{k'} \\ 0 & \mathbf{T}_{m'} \end{bmatrix} \tag{9}$$

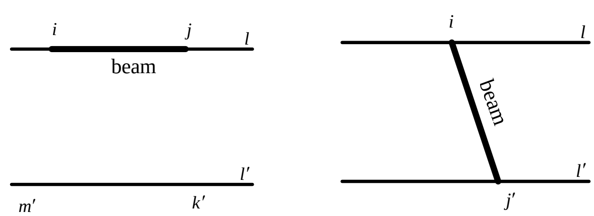


Figure 4: Two Arrangements of Beam Elements in a Strip

It is seen from Figure 4 that beam elements can be located in different ways in a strip, either along one nodal line or across the two nodal lines. In the two cases shown in Figure 4, the displacement vectors and the corresponding transformation matrices are, respectively

$$\boldsymbol{\delta}^e = \{\delta_i, \delta_j\},\ \mathbf{T}_e = \begin{bmatrix} \mathbf{T}_i & 0 \\ \mathbf{T}_j & 0 \end{bmatrix};\ \text{and}\ \ \boldsymbol{\delta}^e = \{\delta_i, \delta_{j'}\},\ \mathbf{T}_e = \begin{bmatrix} \mathbf{T}_i & 0 \\ 0 & \mathbf{T}_{j'} \end{bmatrix} \tag{10}$$

It is noted that Equation (8) is a key equation in the proposed analysis for the element level transformation. By carrying out the standard procedure of the displacement discretelisation in FEM, the original simultaneous equations for nodes will be transformed into the simultaneous equations for the generalised strip displacement.

GENERALISED STIFFNESS MATRIX AND FORCE VECTOR

In the finite element method, the internal virtual work for a stiffened plate with the stiffening beams of three degrees of freedom is given (Zienkiewicz and Taylor 2000) by

$$\delta\boldsymbol{\Pi}_{\text{int}} = \delta\boldsymbol{\Pi}_{P\text{int}} + \delta\boldsymbol{\Pi}_{B\text{int}} = \int_\Omega \delta w\,(\mathbf{L}\nabla)^T\,\mathbf{D}(\mathbf{L}\nabla)w\,d\Omega + \frac{1}{2}\sum_i \int_0^\ell \left(EI\,\frac{\partial^2 w}{\partial s^2} + GJ\,\frac{\partial^2 w}{\partial s \partial n} \right) dx \tag{11}$$

From Equation (8), $w = \mathbf{N}\mathbf{T}_e\mathbf{V}^s$, where $\mathbf{N}$ is the shape function; then, from Equation (11),

$$\overline{\mathbf{K}}^{e}\mathbf{V}^{s} = \bar{\mathbf{f}}^{e} \tag{12}$$

where $\overline{\mathbf{K}}^{e}$ represents the element generalized stiffness matrix with parameters a_{m} and b_{n}, and given by $\overline{\mathbf{K}}^{e} = \overline{\mathbf{K}}_{p}^{e} \oplus \overline{\mathbf{K}}_{b}^{e}$, in which $\overline{\mathbf{K}}_{p}^{e}$ and $\overline{\mathbf{K}}_{b}^{e}$ are the plate and beam stiffness matrices, respectively; and $\oplus$ represents the element assembly. In general, the plate and beam stiffness matrices are given, respectively, by

$$\overline{\mathbf{K}}_{p}^{e} = \mathbf{T}_{e}^{\mathrm{T}}\mathbf{K}_{p}^{e}\mathbf{T}_{e} \quad \text{and} \quad \overline{\mathbf{K}}_{b}^{e} = \mathbf{T}_{e}^{\mathrm{T}}\mathbf{K}_{b}^{e}\mathbf{T}_{e} \tag{13}$$

where $\mathbf{K}_{p}^{e} = \int_{\Omega} [(\mathbf{L}\nabla)\mathbf{N}]^{\mathrm{T}}\mathbf{D}[(\mathbf{L}\nabla)\mathbf{N}]\mathrm{d}\Omega = \int_{\Omega} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathrm{d}\Omega$ with $\mathbf{B} = (\mathbf{L}\nabla)\,\mathbf{N}$, and $\mathbf{K}_{b}^{e} = \mathbf{C}_{b}^{T}\mathbf{K}_{b'}^{e}\mathbf{C}_{b}$

with $$\mathbf{K}_{b'}^{e} = \frac{EI}{l^{3}}\begin{bmatrix} 12 & & & & & \\ 0 & rl^{2} & & & Sym & \\ -6l & 0 & 4l^{2} & & & \\ -12 & 0 & 6l & 12 & & \\ 0 & -rl^{2} & 0 & 0 & rl^{2} & \\ -6l & 0 & 2l^{2} & 6l & 0 & 4l^{2} \end{bmatrix}.$$

Moreover, $\bar{\mathbf{f}}^{e}$ in Equation (12) represents the element generalised force vector and is given by $\bar{\mathbf{f}}^{e} = \mathbf{T}_{e}^{\mathrm{T}}\mathbf{f}^{e}$, where $\mathbf{f}^{e} = \int_{\Omega}\mathbf{N}^{\mathrm{T}}q\mathrm{d}\Omega + \int_{\Gamma}\left(\mathbf{N}_{n}^{\mathrm{T}}\overline{M}_{n} + \mathbf{N}_{s}^{\mathrm{T}}\overline{M}_{ns} + \mathbf{N}^{\mathrm{T}}\overline{S}_{n}\right)\mathrm{d}\Gamma$.

Equation (12) is one of key equations in the proposed analysis. It is used for the element level transformation. By carrying out the standard procedure of the displacement discretelisation in FEM, the original simultaneous equations for nodes will be transformed into the simultaneous equations for the generalised strip displacement.

NUMERICAL STUDIES

Based on the proposed method, a procedure of computation for the analysis of stiffened plates is given as follows:

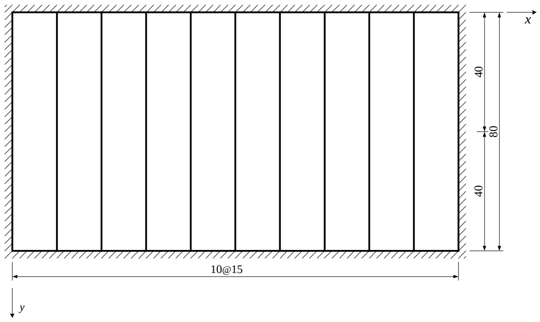
- Divide the plate into several strips by nodal lines;
- Mesh each strip with triangular or quadrilateral elements and develop the element stiffness matrix and force vector;
- Execute transformations;
- Assemble simultaneous equations of the nodal line displacement; then solve the equations;
- Calculate node displacements, then determine the element displacements for plates and beams;
- Determine internal forces of the structure.

In order to demonstrate the accuracy and effectiveness of the proposed method for analysing stiffened plates, three example stiffened plates are considered: (1) rectangular one-way ribbed plate (Song 1986), (2) square plates stiffened by two orthogonal beams (Shen et al. 1987), and (3) square plates stiffened by four orthogonal beams (Shen 1992).

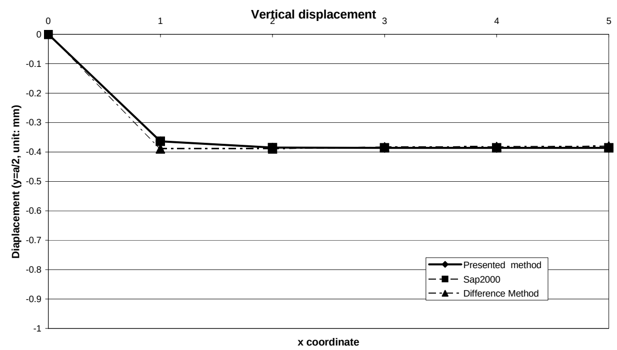
RECTANGULAR ONE-WAY RIBBED PLATE

A rectangular one-way ribbed plate made of aluminium with four fixed edges is shown in Figure 5a, where the stiffening ribs are evenly distributed along the *x*-direction. The plate is subjected to a uniform distributed load with the density of 5.88 N/cm^2. Young's module E = 6.958×10^6 kN/cm^2 and Poission's ratio $\mu = 0.315$. The thickness of the plate t = 0.2 cm, moment of inertia and torsion constant of the ribs are I_b = 3.746 cm^2 and J = 5.236 cm^4.

Figure 5b shows the mid-span vertical displacements of the plate. Comparison of results of vertical displacements is made and given in Table 1. The results from the proposed method agree very well to those from the finite element analysis using a comprehensive FEM package SAP 2000 (Computer and Structures Inc. 1997) and finite difference method.



(a) Ribbed Plate with Fixed Edges (cm)



(b) Vertical Displacement at Mid-Span

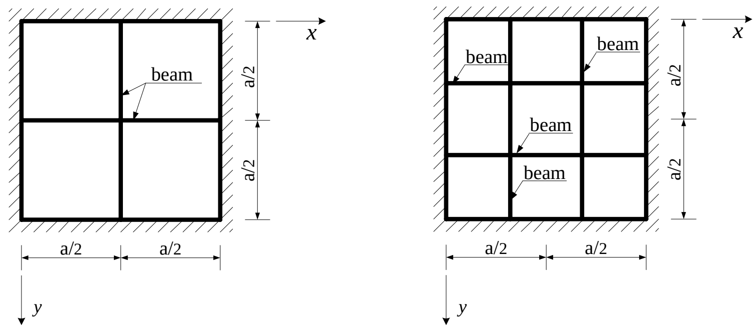
Figure 5: Ribbed Plate with Fixed Edges and Its Vertical Displacement Profile

Table 1: Comparison of Vertical Displacements at Mid-span (Unit: mm)

Coordinate	Proposed	SAP 2000	Finite Diffrence (Song 1986)
(150, 400)	0.3639	0.3639	0.3891
(300, 400)	0.3851	0.3857	0.3891
(450, 400)	0.3860	0.3861	0.3827
(600, 400)	0.3863	0.3860	0.3818
(750, 400)	0.3863	0.3860	0.3803

SQUARE PLATE STIFFENED BY TWO AND FOUR ORTHOGONAL BEAMS

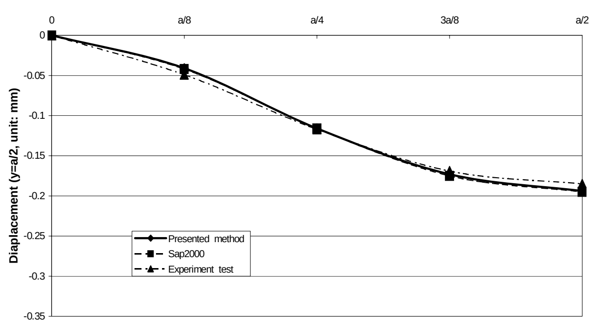
Two fixed-edge square plates, which are stiffened by two and four orthogonal beams respectively, are shown in Figure 7. Both plates are subjected to a uniform distributed load with the density of 4.8 N/cm^2. The side length and thickness of the plates a = 20.18 cm and t = 0.282 cm. The moment of inertia for the beams in the plate stiffened by two orthogonal beams (Plate 1, Figure 7a) $I_b = 5.96 \times 10^{-3}$ cm^4 and torsion constant $J = 9.96 \times 10^{-3}$ cm, and in the plate stiffened by four orthogonal beams (Plate 2, Figure 7b) $I_b = 4.51 \times 10^{-3}$ cm^4 and torsion constant J = 8.61 × 10^{-3} cm. The plates and beams are made of the same material; Young's module E = 20.737 × 10^{6} N/cm^2, and Poisson ratio μ = 0.3.



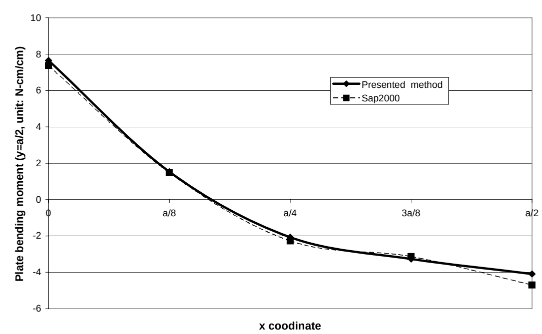
(a) (b)

Figure 7: Square Stiffened Plates with Fixed edges (a = 20.18 cm). (a) Stiffened by Two Orthogonal Beams; (b) Stiffened by Four Orthogonal Beams

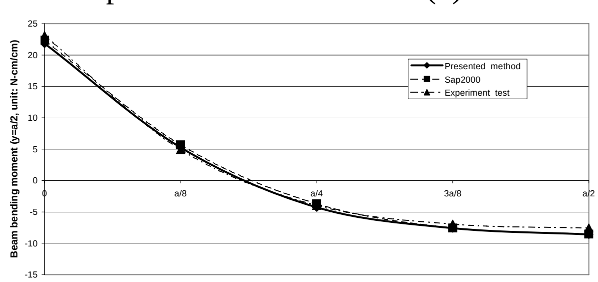
The mid-span vertical displacements and bending moments of the plates and beams for the two stiffened plates are shown in Figures 8 and 9. Comparisons of results, which include (1) vertical displacements and bending moments of plate and beam for Plate 1, and (2) vertical displacements and bending moment of Plate 2, by different methods are made and presented in Tables 2 to 4. It is seen from these figures and tables that the results of the proposed method show very good agreement with those of the finite element analysis and experiment.



(a) Plate vertical displacement

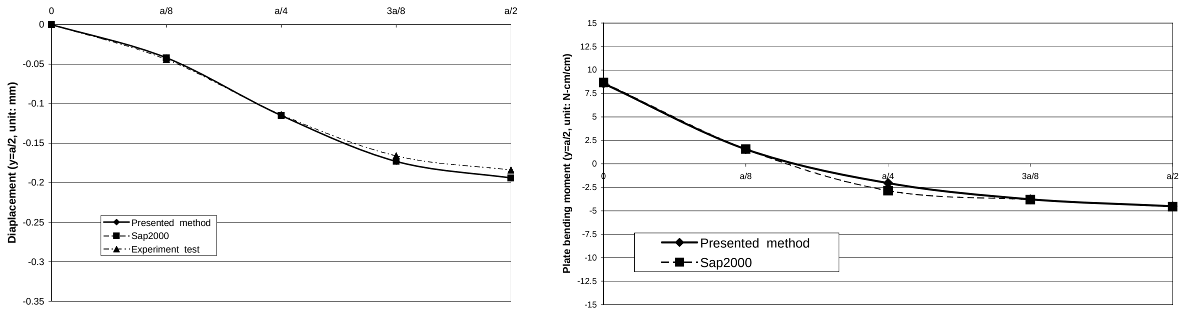


(b) Plate bending moment ($M_{x\text{-}x}$)



(c) Beam Bending Moment

Figure 8: Displacements and Bending Moments of Plate Stiffened by Two Orthogonal Beams along Symmetrical Axis y = a/2



(a) Plate Vertical Displacement (b) Plate Bending Moment (M_{x-x})

Figure 9: Displacements and Bending Moments of Plate Stiffened by Four Orthogonal Beams along Symmetrical Axis $y = a/2$

Table 2: Comparison of Vertical Displacements along Symmetrical Axis from for Plate Stiffened by Two Orthogonal Beams (Unit: mm)

Coordinate (a = 201.8)	Proposed	SAP 2000	Experiment (Shen et al 1987)
(0, a/2)	0.000	0.000	0.000
(a/8, a/2)	0.041	0.042	0.049
(a/4, a/2)	0.116	0.116	0.117
(3a/8, a/2)	0.173	0.175	0.169
(a/2, a/2)	0.194	0.195	0.185

CONCLUSIONS

In this paper, the stiffened plate analogy is presented to model the plate-and-grid structures. A semi-analytical method is proposed based on the finite element method to obtain an accurate analysis of the stiffened plates. The proposed method is used effectively and conveniently to solve the problem of a stiffened plate with irregular shape and openings, multi-connective domain or discontinuous material property and arbitrary oriented stiffeners, but involving much less unknown variables as compared to the general finite element methods. Moreover, it is shown that the proposed method can have more flexible geometry adaptability than the finite strip method, higher efficiency in reducing the unknown variables than the boundary element method, and much less computation time and cost can be achieved by employing analytical transformations instead of the traditional matrix inverse. Moreover, although the proposed approach is based on the finite element method, only a few modifications to the original FEM programming are needed when immigrating the FEM to proposed method.

It has been shown in the numerical examples that proposed method provides an efficient, yet accurate, means of analysing general stiffened plates. Results of the proposed analyses for the example structures show very good agreement with those of the FEM and experiment.

Table 3: Comparison of Bending Moments of Plate and Beam along Symmetrical Axis for Plate Stiffened by Two Orthogonal Beams (Experiment by Shen et al 1987)

Coordinate (*a* = 201.8 mm)	Plate (N-cm/cm)			Beam (N-cm)		
	Proposed	SAP 2000	Experiment	Proposed	SAP 2000	Experiment
(0, *a*/2)	-7.652	-7.370	-7.927	-21.828	-22.345	-23.080
(*a*/8, *a*/2)	-1.533	-1.479	-	-5.263	-5.683	-4.919
(a/4, a/2)	2.072	2.273	-	4.264	3.704	3.973
(3*a*/8, *a*/2)	3.266	3.133	-	7.585	7.563	6.950
(*a*/2, *a*/2)	4.090	4.699	3.379	8.596	8.537	7.568

Table 4: Comparison of Vertical Displacement and Bending Moments of Plate along Symmetrical Axis for Plate Stiffened by Four Orthogonal Beams (Experiment by Shen 1992)

Coordinate (*a* = 201.8 mm)	Displacement (mm)			Bending Moment (N-cm/cm)		
	Proposed	SAP 2000	Experiment	Proposed	SAP 2000	Experiment
(0, *a*/2)	0.000	0.000	0.000	-8.554	-8.669	-8.230
(*a*/8, *a*/2)	0.042	0.042	0.044	-1.559	-1.570	-
(a/4, a/2)	0.115	0.115	0.114	2.053	2.858	-
(3*a*/8, *a*/2)	0.173	0.173	0.166	3.773	3.811	-
(*a*/2, *a*/2)	0.194	0.194	0.184	4.513	4.564	3.491

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