http ://luth.obspm.fr/~luthier/nottale/

QUANTUM MECHANICS

Effects on the equations of motion
of the fractal structures of the geodesics of a nondifferentiable space

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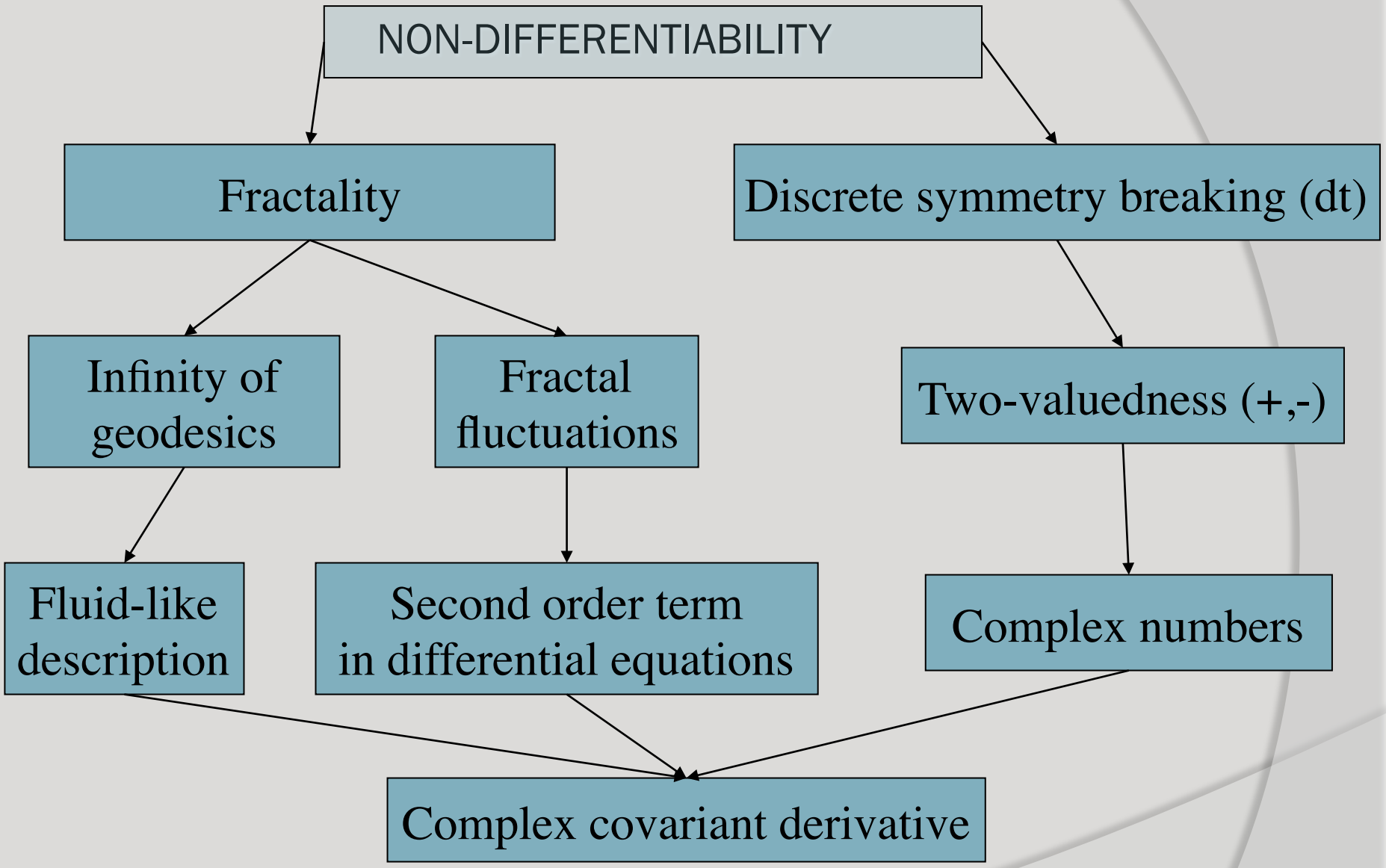


$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta$$

Dilatation operator (Gell-Mann-Lévy method):

$$\mathcal{L}(\varepsilon') = \mathcal{L}(\varepsilon + \varepsilon\, d\rho) = \mathcal{L}(\varepsilon) + \frac{\partial \mathcal{L}(\varepsilon)}{\partial \varepsilon}\, \varepsilon\, d\rho = (1 + \tilde{D}\, d\rho)\, \mathcal{L}(\varepsilon),$$

$$\tilde{D} = \varepsilon\, \frac{\partial}{\partial \varepsilon} = \frac{\partial}{\partial \ln \varepsilon}$$

FIRST ORDER SCALE DIFFERENTIAL EQUATION:

$$\frac{\partial \mathcal{L}(x, \varepsilon)}{\partial \ln \varepsilon} = \beta(\mathcal{L})$$

Taylor expansion:

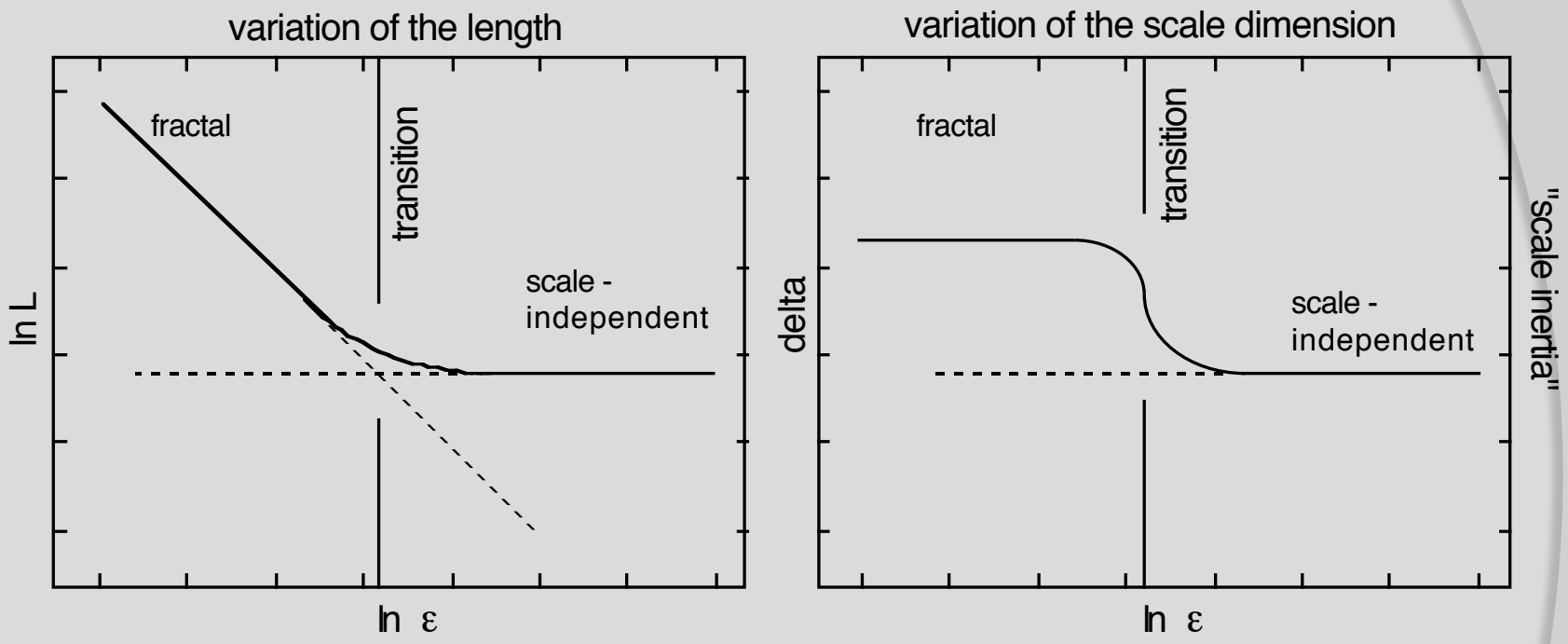
$$\frac{\partial \mathcal{L}(x, \varepsilon)}{\partial \ln \varepsilon} = a + b\mathcal{L}$$

Solution: fractal of constant dimension + transition:

$$\mathcal{L}(x, \varepsilon) = \mathcal{L}_0(x) \left[1 + \zeta(x) \left(\frac{\lambda}{\varepsilon}\right)^{-b}\right]$$

$$\mathcal{L}(x,\varepsilon) = \mathcal{L}_0(x)\left[1+\left(\frac{\lambda}{\varepsilon}\right)^{\delta}\right]$$

$$\delta_{\text{eff}} = \frac{\delta}{1+\left(\frac{\varepsilon}{\lambda}\right)^{\delta}}$$



Dependence on scale of the length (=fractal coordinate) and of the effective fractal dimension

$$\delta = \frac{d\ln\mathcal{L}}{d\ln(\lambda/\varepsilon)} \quad = \; D_{\mathrm{F}} - D_{\mathrm{T}}$$

Case of « scale-inertial » laws (which are solutions of a first order scale differential equation in scale space).

« Galileo » scale transformation group

Asymptotic behavior:

$$\mathcal{L}(x,\varepsilon) = \mathcal{L}_0(x)\left(\frac{\lambda}{\varepsilon}\right)^{\delta}$$

Scale transformation: $\varepsilon \longrightarrow \varepsilon'$

$$\ln\frac{\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln\frac{\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \delta(\varepsilon)\ln\left(\frac{\varepsilon}{\varepsilon'}\right)$$

$$\delta(\varepsilon') = \delta(\varepsilon)$$

Law of composition of dilatations: $\varepsilon \longrightarrow \varepsilon' \longrightarrow \varepsilon''$

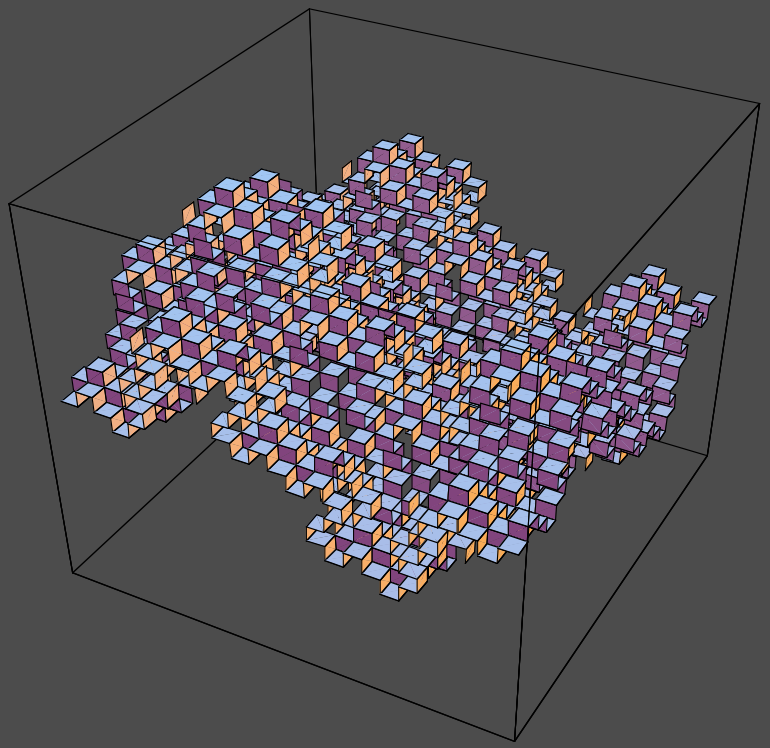
$$\ln\left(\frac{\varepsilon''}{\varepsilon}\right) = \ln\left(\frac{\varepsilon''}{\varepsilon'}\right) + \ln\left(\frac{\varepsilon'}{\varepsilon}\right)$$

Result: mathematical structure of a Galileo group —>

-comes under the principle of relativity (of scales)-

$$X' = X - VT,\ \ T' = T,\ \ W = U + V$$

Road toward Schrödinger (1): infinity of geodesics



—> generalized « fluid » approach:

$$V(t) \rightarrow V[x(t), t, \delta t] = v[x(t), t] + W[x(t), t, \delta t]$$

Differentiable

Non-differentiable

Road toward Schrödinger (2): 'differentiable part' and 'fractal part'

Minimal scale law (in terms of the *space* resolution):

$$\mathcal{L}(x,\varepsilon) = \mathcal{L}_0(x)\left[1+\left(\frac{\lambda}{\varepsilon}\right)^{\delta}\right] \qquad (\varepsilon_x/\lambda)^{D_F} = (\varepsilon_t/\lambda)$$

Differential version (in terms of the *time* resolution):

$$dX = dx + d\xi$$

$$dx = v\,dt$$

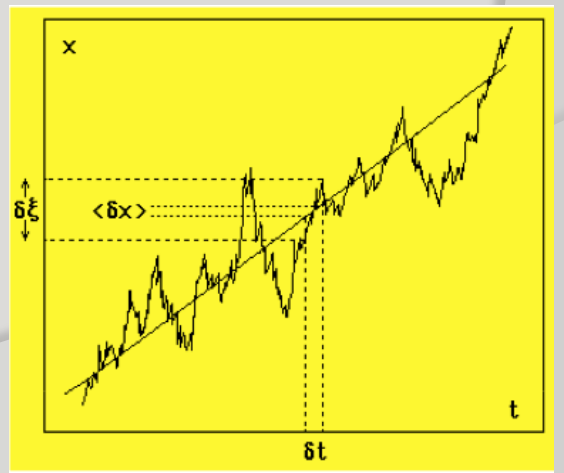
$$d\xi = \eta\sqrt{2\mathcal{D}}(dt^2)^{1/2D_F}$$

Stochastic variable:

$$<\eta>=0 \qquad <\eta^2>=1$$

Case of the critical fractal dimension $D_F = 2$:

$$d\xi = \eta\sqrt{2\mathcal{D}}dt^{1/2}$$



Road toward Schrödinger (3):
non-differentiability –––> complex numbers

Standard definition of derivative

$$\frac{df}{dt} = \lim_{dt\to 0} \frac{f(t+dt) - f(t)}{dt} = \lim_{dt\to 0} \frac{f(t) - f(t-dt)}{dt}$$

DOES NOT EXIST ANY LONGER —> new definition

$f(t,dt)$ = fractal fonction (equivalence class, cf LN93)

Explicit fonction of dt = scale variable (generalized « resolution »)

$$f'_+(t,dt) = \frac{f(t+dt,dt) - f(t,dt)}{dt}$$

$$f'_-(t,dt) = \frac{f(t,dt) - f(t-dt,dt)}{dt}$$

TWO definitions instead of one: they transform one in another by the reflection (dt <—> $-dt$)

$$dX_+(t) = v_+\, dt + d\xi_+(t)$$

$$dX_-(t) = v_-\, dt + d\xi_-(t)$$

$$\mathcal{V} = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}$$

Covariant derivative operator

$$\frac{d\!\!\!^{-}}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right)$$

Classical (differentiable) part

$$\mathcal{V} = \frac{d\!\!\!^{-}}{dt}x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dX_i}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt}$$

$$< \frac{dX_i}{dt} >= \frac{dx_i}{dt} \qquad < \frac{dX_i dX_j}{dt} >=< \frac{d\xi_i d\xi_j}{dt} >= 2\mathcal{D}\,\delta_{ij}$$

$$\frac{d_\pm f}{dt} = \left(\frac{\partial}{\partial t} + v_\pm.\nabla \pm \mathcal{D}\,\Delta\right) f$$

$$\boxed{\frac{d\!\!\!^{-}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta}$$

Improvement of « quantum » covariance

Ref.: Nottale L., 2004, American Institute of Physics Conference Proceedings 718, 68-95
"The Theory of Scale Relativity : Non-Differentiable Geometry and Fractal Space- Time".
http ://luth.obspm.fr/~luthier/nottale/arcasys03.pdf

Introduce complex velocity operator:

$$\widehat{\mathcal{V}} = \mathcal{V} - iD\nabla$$

New form of covariant derivative:

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + \widehat{\mathcal{V}}.\nabla$$

satisfies first order Leibniz rule for partial derivative and law of composition (see also Pissondes' s work on this point)

FRACTAL SPACE-TIME-->QUANTUM MECHANICS

Covariant derivative operator

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta$$

Fundamental equation of dynamics

$$\frac{d'\mathcal{V}}{dt} = -\nabla\phi$$

Change of variables (S = complex action) and integration

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t)dt \qquad \psi = e^{i\mathcal{S}/2m\mathcal{D}}$$

Generalized Schrödinger equation

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{1}{2}\phi\,\psi = 0$$

Ref: LN, 93-04, Célérier & LN 04,07. See also works by: Ord, Hermann, Pissondes, Dubois, Jumarie, Cresson, Ben Adda, Agop, …

Hamiltonian: covariant form

$$\mathcal{P} = -i\mathcal{S}_0 \nabla \ln \psi \qquad \mathcal{P}\psi = -i\hbar \nabla \psi$$

$$\mathcal{L} = \frac{\widehat{d}\,\mathcal{S}}{dt} = \frac{\partial \mathcal{S}}{\partial t} + \widehat{\mathcal{V}}.\nabla \mathcal{S}$$

$$\mathcal{P} = \nabla \mathcal{S} \qquad \mathcal{H} = -\partial \mathcal{S}/\partial t$$

$$\mathcal{H} = \widehat{\mathcal{V}}.\mathcal{P} - \mathcal{L}$$

—>

$$\mathcal{H} = \mathcal{V}.\mathcal{P} - i\mathcal{D}\nabla.\mathcal{P} - \mathcal{L}$$

Additional energy term specific of quantum mechanics: explained here as manifestation of nondifferentiability and strong covariance

Newton

$$m\frac{d}{dt}\mathcal{V} = -\nabla\Phi$$

$$\mathcal{V} = \nabla\mathcal{S}/m$$

$$\psi = e^{i\mathcal{S}/\mathcal{S}_0}$$

$$i\mathcal{S}_0\frac{d}{dt}(\nabla \ln\psi) = \nabla\Phi$$

$$\mathcal{V} = -i\frac{\mathcal{S}_0}{m}\nabla(\ln\psi)$$

$$\nabla\Phi = i\mathcal{S}_0\left(\frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta\right)(\nabla \ln\psi)$$

$$\nabla\Phi = i\mathcal{S}_0\left[\frac{\partial}{\partial t}\nabla \ln\psi - i\left\{\frac{\mathcal{S}_0}{m}(\nabla \ln\psi.\nabla)(\nabla \ln\psi) + \mathcal{D}\Delta(\nabla \ln\psi)\right\}\right]$$

$$\frac{d}{dt}\mathcal{V} = -2\mathcal{D}\nabla\left\{i\frac{\partial}{\partial t}\ln\psi + \mathcal{D}\frac{\Delta\psi}{\psi}\right\} = -\nabla\Phi/m$$

$$\mathcal{S}_0 = 2m\mathcal{D}$$

Schrödinger

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\Phi}{2m}\psi = 0$$

Newton

$$m\frac{d}{dt}\mathcal{V} = -\nabla\Phi$$

$$\mathcal{V} = \nabla\mathcal{S}/m$$

$$\psi = e^{i\mathcal{S}/\mathcal{S}_0}$$

$$i\mathcal{S}_0\frac{d}{dt}(\nabla \ln\psi) = \nabla\Phi$$

$$\mathcal{V} = -i\,\frac{\mathcal{S}_0}{m}\,\nabla(\ln\psi)$$

$$\nabla\Phi = i\mathcal{S}_0\left(\frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta\right)(\nabla \ln\psi)$$

$$\nabla\Phi = i\mathcal{S}_0\left[\frac{\partial}{\partial t}\nabla \ln\psi - i\left\{\frac{\mathcal{S}_0}{m}(\nabla \ln\psi.\nabla)(\nabla \ln\psi) + \mathcal{D}\Delta(\nabla \ln\psi)\right\}\right]$$

$$\frac{d}{dt}\mathcal{V} = -2\mathcal{D}\nabla\left\{i\frac{\partial}{\partial t}\ln\psi + \mathcal{D}\frac{\Delta\psi}{\psi}\right\} = -\nabla\Phi/m$$

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\Phi}{2m}\psi = 0$$

$$\mathcal{S}_0 = 2m\mathcal{D}$$

Schrödinger

$$H\psi = i\,2m\mathcal{D}\,\frac{\partial\psi}{\partial t}$$

Origin of complex numbers in quantum mechanics. 1.

Two valuedness of the velocity field —> need to define a new product: algebra doubling A—>A^2

General form of a bilinear product :

$$c^k = a^i \, \omega_{ij}^k \, b^j$$

i,j,k = 1,2 —> new product defined by the 8 numbers ω_{ij}^k

Recover the classical limit —> A subalgebra of A^2

Then (a,0)=a. We define (0,1)=α and therefore only 2 coefficients are needed:

$$\alpha^2 = \omega_0 + \omega_1 \alpha$$

Complex numbers. Origin. 2.

Define the new velocity doublet, including the divergent (explicitly scale-dependent) part:

$$\tilde{\mathcal{V}} = \mathcal{V} + \mathcal{W} = \left(\frac{v_+ + v_-}{2} - \alpha \frac{v_+ - v_-}{2} \right) + \left(\frac{w_+ + w_-}{2} - \alpha \frac{w_+ - w_-}{2} \right)$$

Full Lagrange function (Newtonian case):

$$\mathcal{L} = \frac{1}{2} m \langle (\mathcal{V} + \mathcal{W})^2 \rangle = \frac{1}{2} m \left(\langle \mathcal{V}^2 \rangle + \langle \mathcal{W}^2 \rangle \right)$$

Infinite term in the Lagrangian ?

$$4 \langle \mathcal{W}^2 \rangle = \langle [(w_+ + w_-) - \alpha (w_+ - w_-)]^2 \rangle$$
$$= \langle (w_+^2 + w_-^2)(1 + \alpha^2) - 2\alpha(w_+^2 - w_-^2) + 2 w_+ w_- (1 - \alpha^2) \rangle$$

Since $\langle w_+^2 \rangle - \langle w_-^2 \rangle = 0$ and $\langle w_+ w_- \rangle = 0$

—> Infinite term suppressed in the Lagrangian provided:

$$\alpha^2 + 1 = 0 \Rightarrow \alpha = \pm i$$

QED

General relativity:
covariant derivative

$$DA^k = dA^k + \Gamma^k_{ij} A^i dx^j$$

Scale relativity:
covariant derivative

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \mathcal{V}\cdot\nabla - i\,\mathcal{D}\,\triangle$$

Geodesics equation:

$$\frac{Du^k}{ds} = \frac{du^k}{ds} + \Gamma^k_{ij} u^i u^j = 0$$

Geodesics equation:

$$\frac{d'\,\mathcal{V}^k}{dt} = 0$$

Newtonian approximation:

$$\frac{DV}{dt} = \frac{dV}{dt} + \nabla\left(\frac{\phi}{m}\right) = 0$$

Quantum form

$$\frac{d'\,\mathcal{V}}{dt} = -2\,\nabla\left(\frac{\mathcal{D}^2\,\Delta\psi + i\mathcal{D}\,\partial\psi/\partial t}{\psi}\right) = 0$$

General + scale relativity

$$\frac{\bar{D}}{dt}\,\mathcal{V} = \frac{d'\,\mathcal{V}}{dt} + \nabla\left(\frac{\phi}{m}\right) = 0 \quad \Rightarrow \quad \mathcal{D}^2\,\triangle\,\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\phi}{2m}\,\psi = 0$$

Three representations

Geodesical (U,V)

$$\frac{d\,\mathcal{V}}{dt} = -\nabla\left(\frac{\phi}{m}\right)$$

Generalized Schrödinger (P,θ)

$$\mathcal{D}^2 \triangle \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\psi$$

Euler + continuity (P, V)

$$m\,(\frac{\partial}{\partial t} + V\cdot\nabla)V = -\nabla(\phi + Q)$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0$$

New « potential » energy:

$$Q = -2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}$$

Five representations (forms) of ScR equations

(1) Fondamental eq. of dynamics/ geodesic eq.

$$\mathcal{V} = V - i\,U$$

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta$$

$$m\,\frac{\widehat{d}\,\mathcal{V}}{dt} = -\nabla\phi$$

(2) Schrödinger $\mathcal{V} = -2\,i\,\mathcal{D}\nabla \ln\psi$

$$\mathcal{D}^2\,\triangle\,\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\,\psi$$

(3) Fluid mechanics($P = |\psi|^2$, V) -> continuity + Euler + quantum potential

$$\left(\frac{\partial}{\partial t} + V.\nabla\right) V = -\nabla\left(\frac{\phi}{m} - 2\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}\right)$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0$$

(4) Coupled bi-fluid (U, V)

$$\left(\frac{\partial}{\partial t} + V.\nabla\right) U = -(U.\nabla + \mathcal{D}\Delta)V$$

$$\left(\frac{\partial}{\partial t} + V.\nabla\right) V = (U.\nabla + \mathcal{D}\Delta)U - \nabla\left(\frac{\phi}{m}\right)$$

(5) Diffusion (v_+, v_-) Fokker-Planck + BFP

$$\mathcal{V} = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}$$

$$v_+ - v_- = 2\,\mathcal{D}\nabla \ln P$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(Pv_+) = \mathcal{D}\Delta P$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(Pv_-) = -\mathcal{D}\Delta P$$

7 significations (and measurement methods) of coefficient $\mathcal{D}$

* Diffusion coefficient

$$< d\xi^2 >= 2\mathcal{D}dt \Rightarrow \mathcal{D} = \frac{< d\xi^2 >}{2dt}$$

- Amplitude of fractal fluctuations

* Generalized Compton length

$$\lambda_c = \frac{2\mathcal{D}}{c} \Rightarrow \mathcal{D} = \frac{1}{2} c\, \lambda_c$$

Generalized de Broglie length

$$\lambda_{\mathrm{dB}} = \frac{2\mathcal{D}}{< v >} \Rightarrow \mathcal{D} = \frac{1}{2} < v > \lambda_{\mathrm{dB}}$$

- Generalized thermal de Broglie length

$$\lambda_{\mathrm{th}} = \frac{2\mathcal{D}}{\sqrt{< v^2 >^{1/2}}} = \mathcal{D}\sqrt{\frac{2m}{k_B T}}$$

* Heisenberg relation (x,v)

$$\mathcal{D} = \Delta x \times \Delta v$$

Heisenberg relation (t,v)

$$\mathcal{D} = \Delta t \times \Delta E/m = \Delta t \times \Delta(v^2)$$

*Energy quantization

$$E = 2m\mathcal{D}\omega \Rightarrow \mathcal{D} = \frac{E}{2m\omega}$$

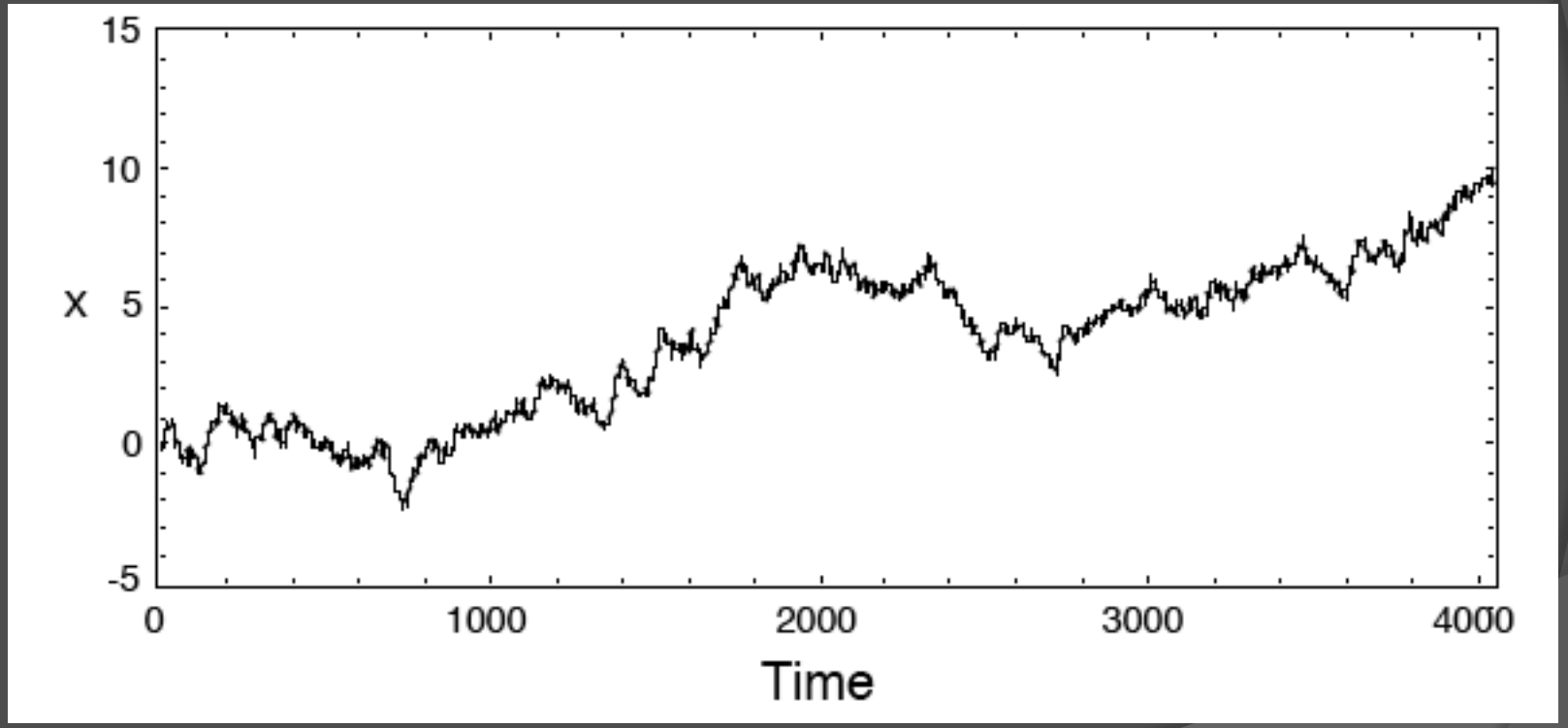
etc…

SOLUTIONS
Visualizations, simulations

Geodesics stochastic differential equations

$$
\begin{aligned}
dx &= v_+ dt + d\xi_+, \quad \frac{\partial P}{\partial t} + \mathrm{div}(Pv_+) = \mathcal{D}\Delta P : \text{ Fokker} - \text{Planck.} \\
dx &= v_- dt + d\xi_-, \quad \frac{\partial P}{\partial t} + \mathrm{div}(Pv_-) = -\mathcal{D}\Delta P : \text{ no classical process.} \\
dx &= V dt + d\xi_V, \quad \frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0 : \text{ continuity equation.} \\
dx &= U dt + d\xi_U, \quad \mathrm{div}(PU) = \mathcal{D}\Delta P : \text{ no classical process.} \qquad (1
\end{aligned}
$$

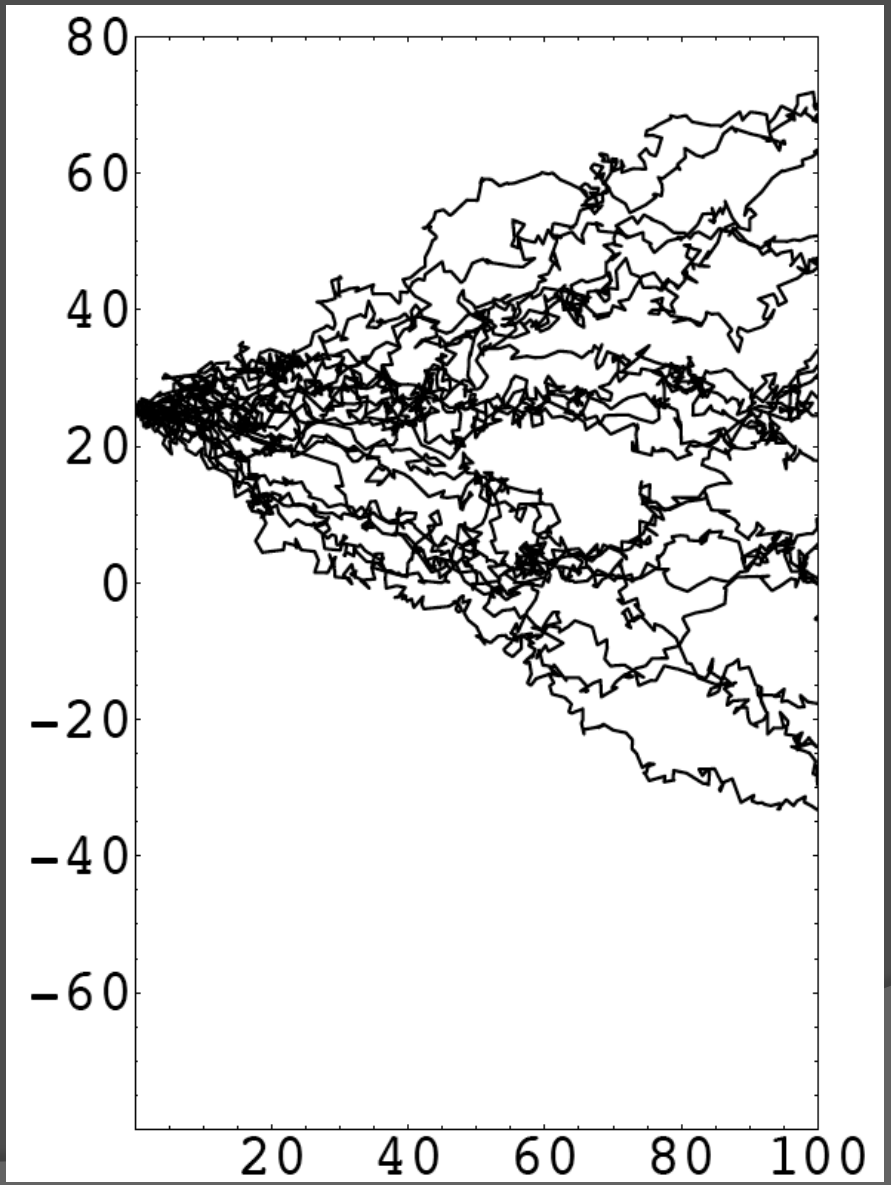
Numerical simulation of fractal geodesics: free particle



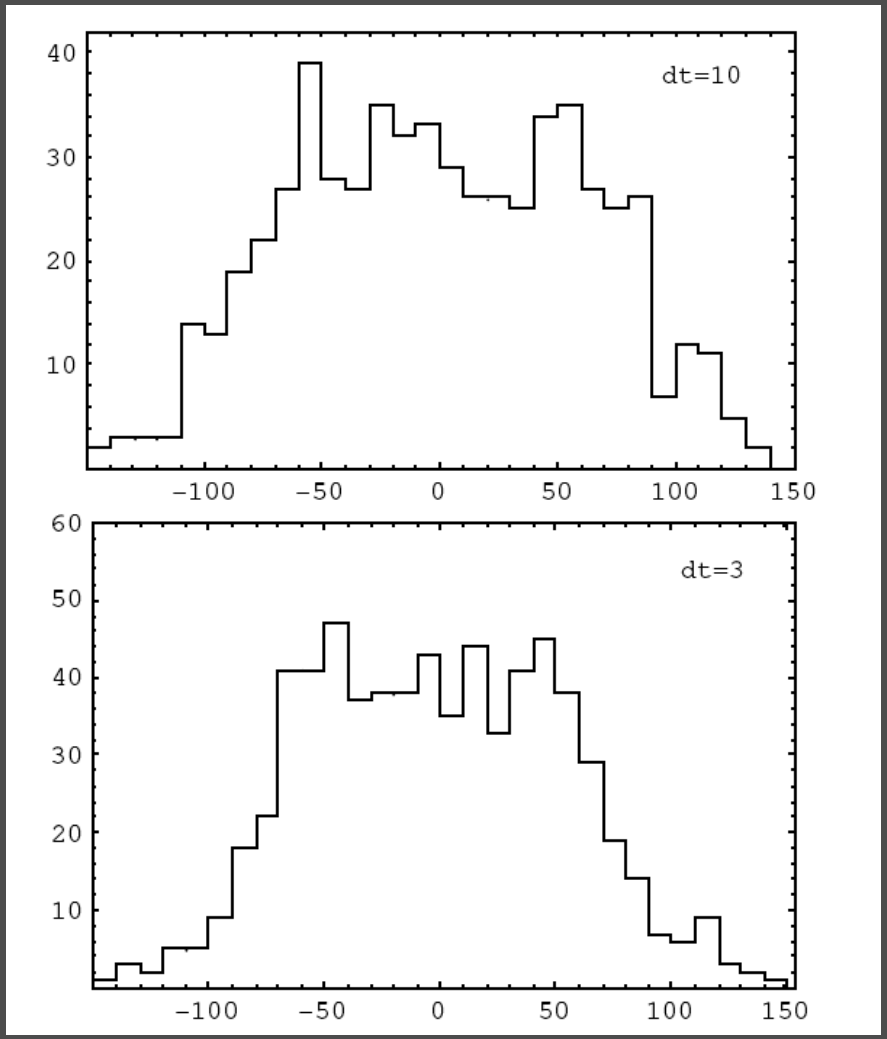
Cf R. Hermann 1997 J Phys A

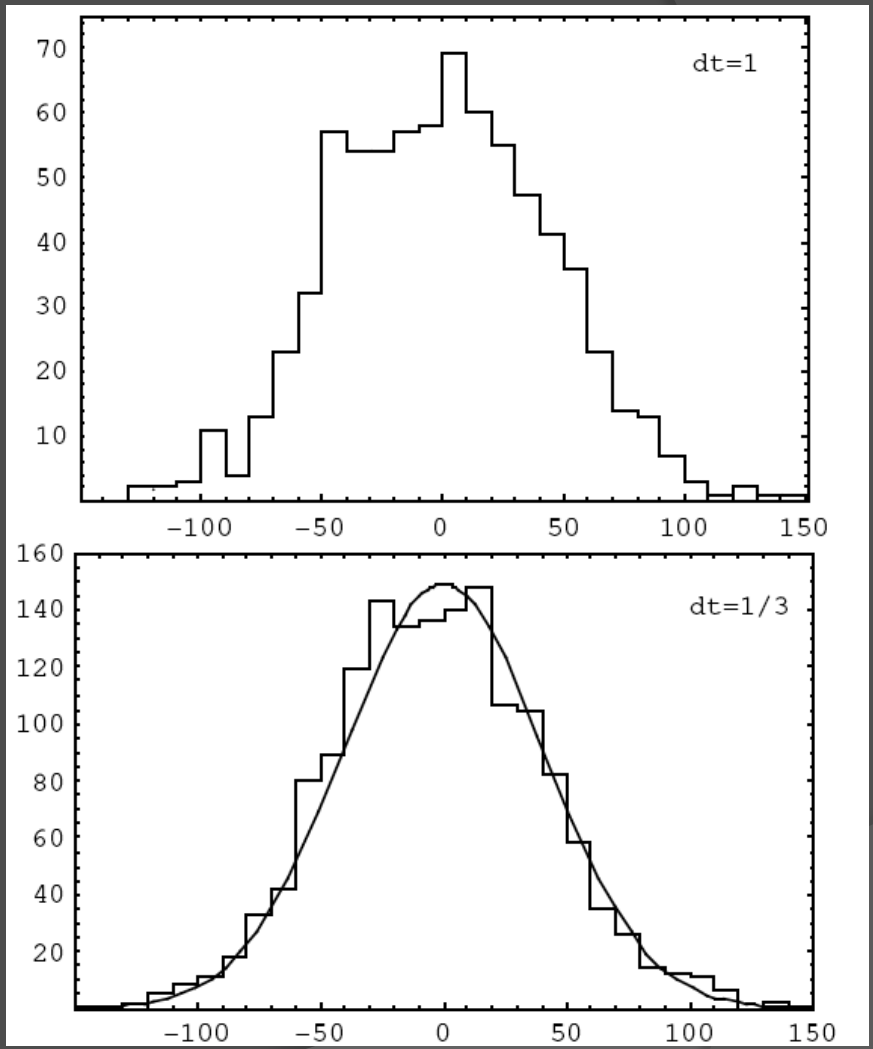
Young hole experiment: one slit

Simulation
of
geodesics



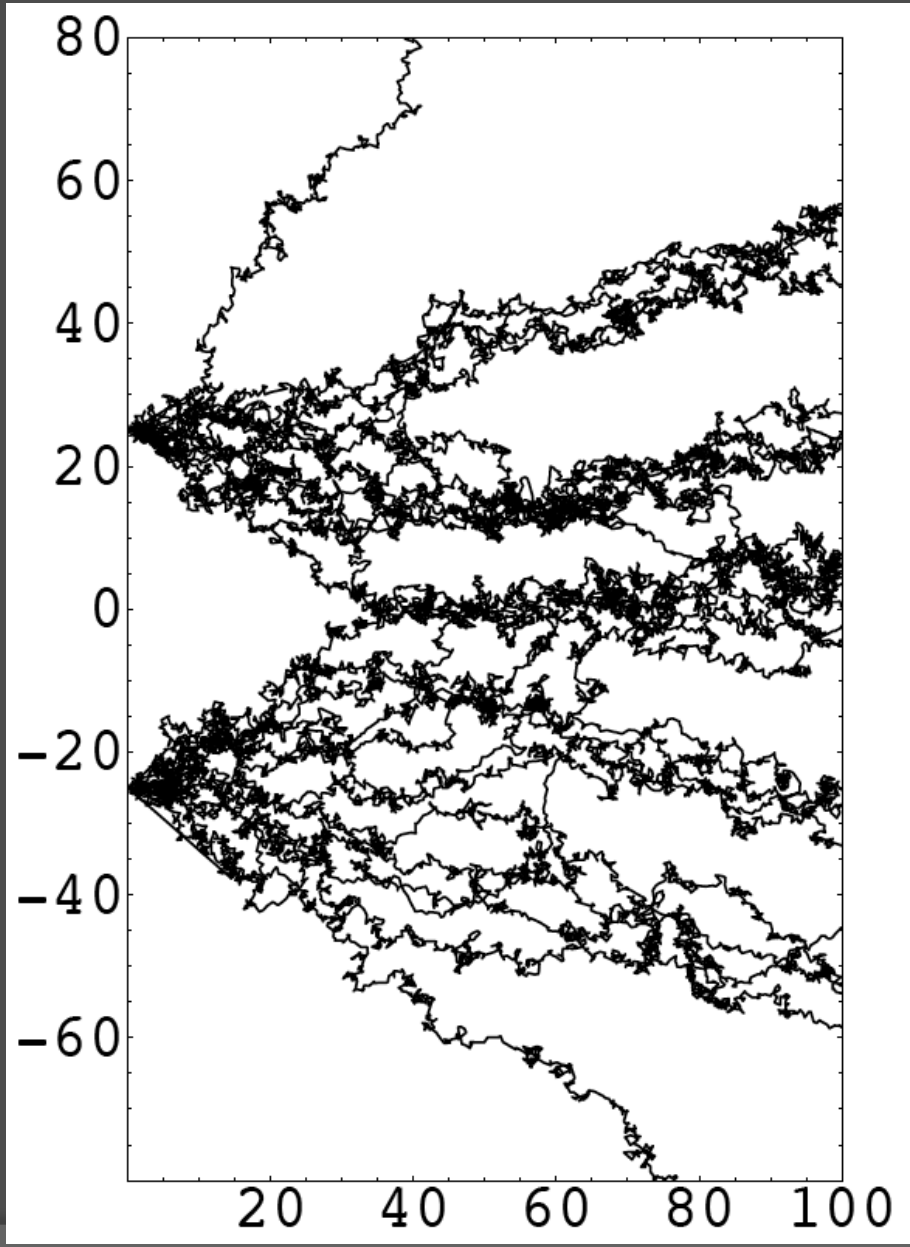
Young hole experiment: one slit



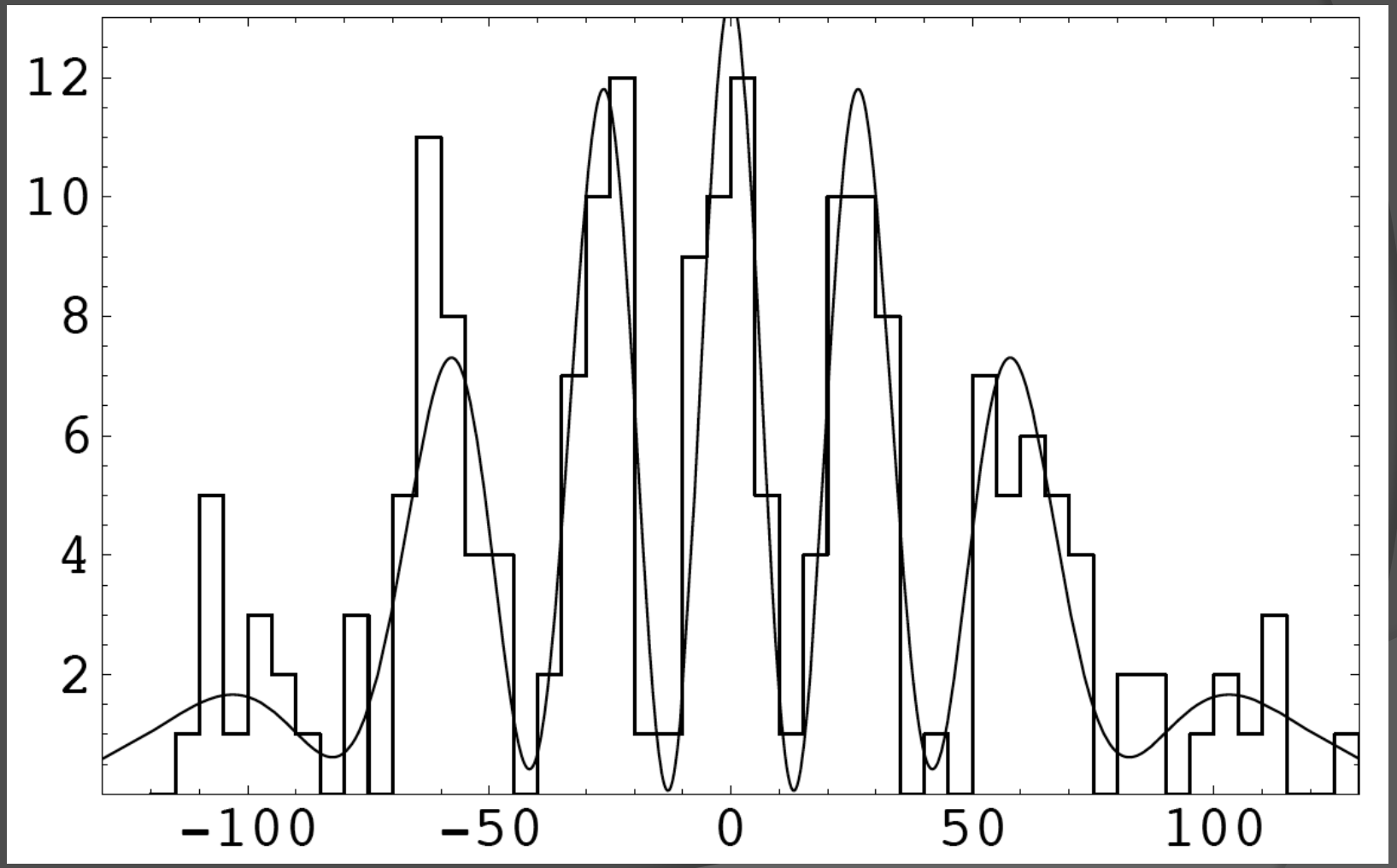


Scale dependent simulation: quantum-classical transition

Young holes: 2 slits

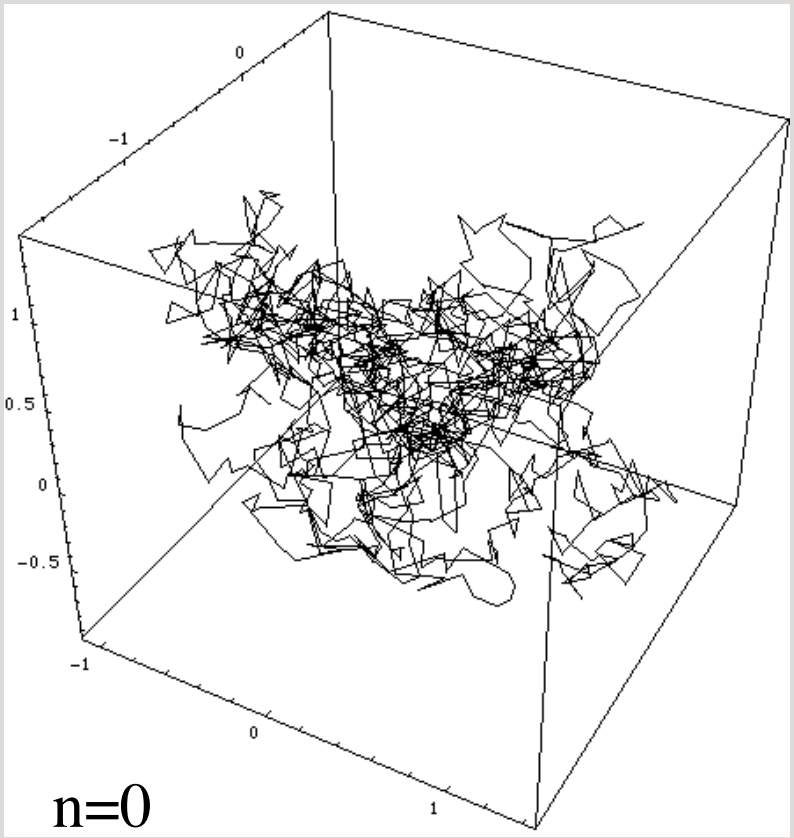


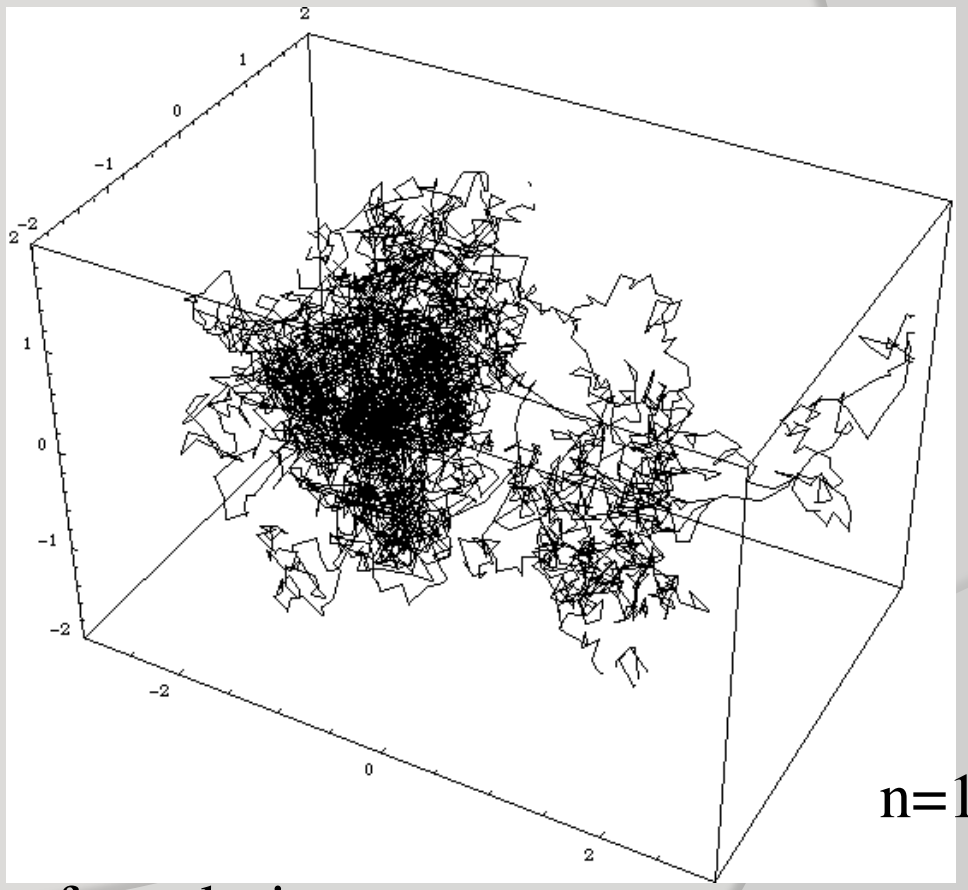
Young hole experiment: two-slit



3D isotropic harmonic oscillator

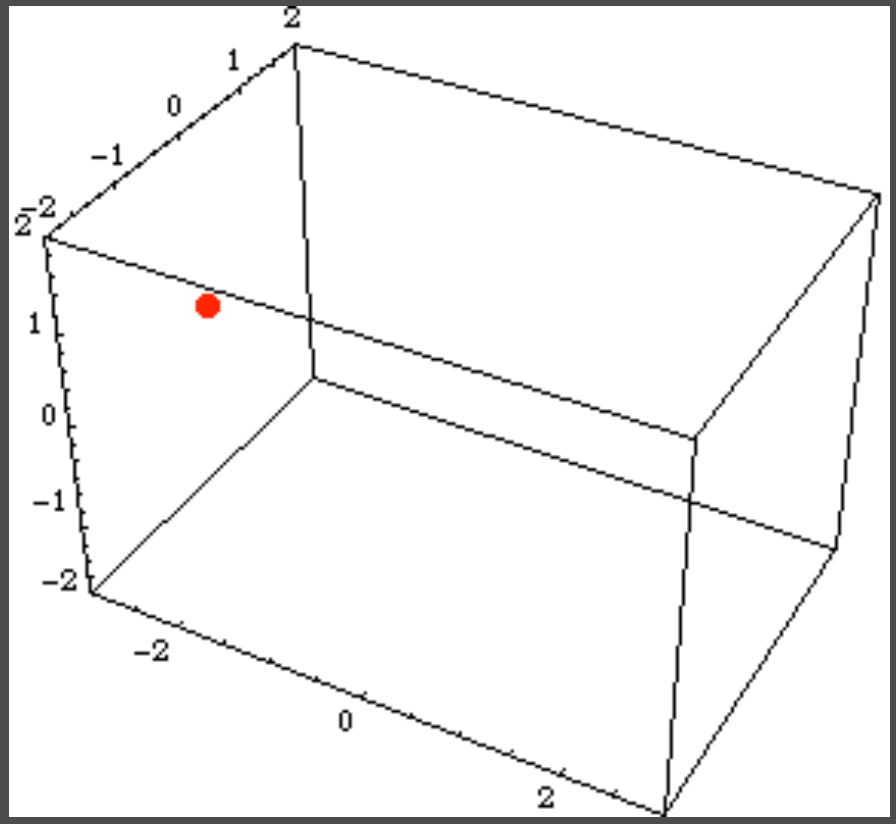
simulation of process $dx^k = v^k_+ \, dt + d\xi^k_+$





Examples of geodesics

3D isotropic harmonic oscillator potential

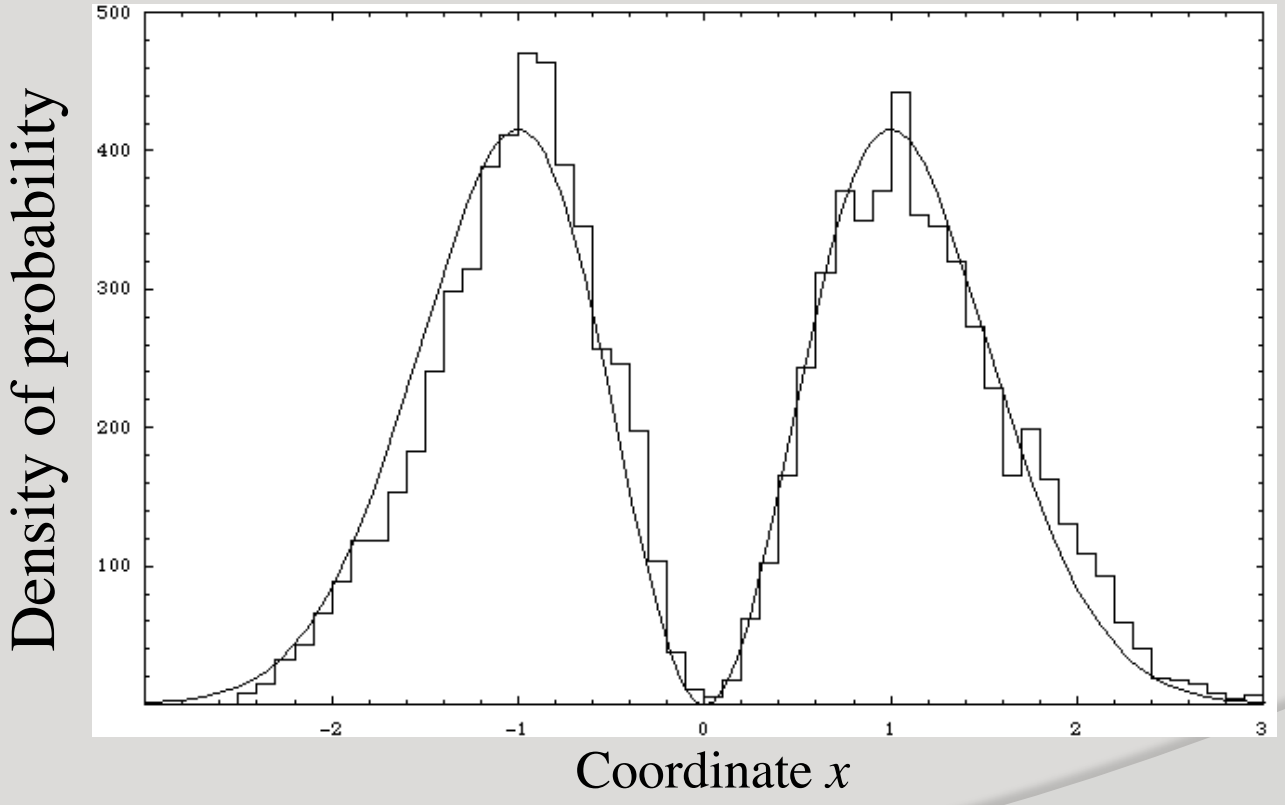


First excited level : simulation of the process $dx = v_+\, dt + d\xi_+$

Animation

3D isotropic harmonic oscillator potential

First excited level: simulation of process $dx = v_+\, dt + d\xi_+$

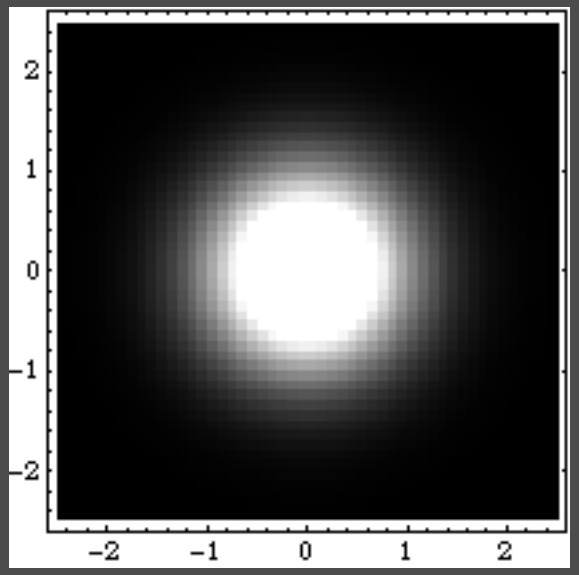


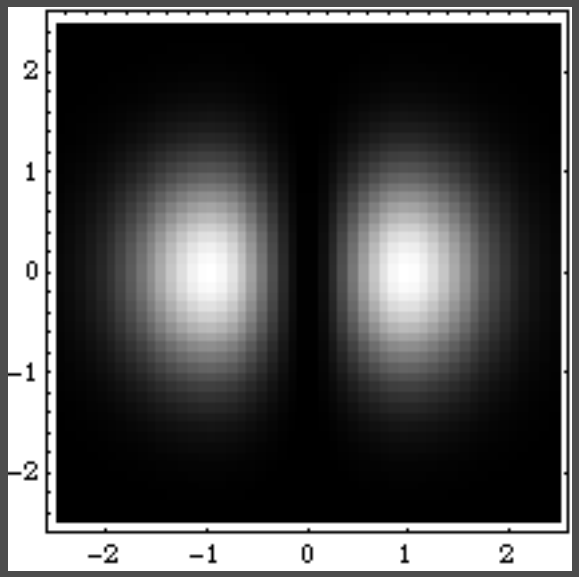
Comparaison simulation - QM prediction: 10000 pts, 2 geodesics

Solutions: 3D harmonic oscillator potential 3D (constant density)

$$E = (3+2n)\, mD\omega$$

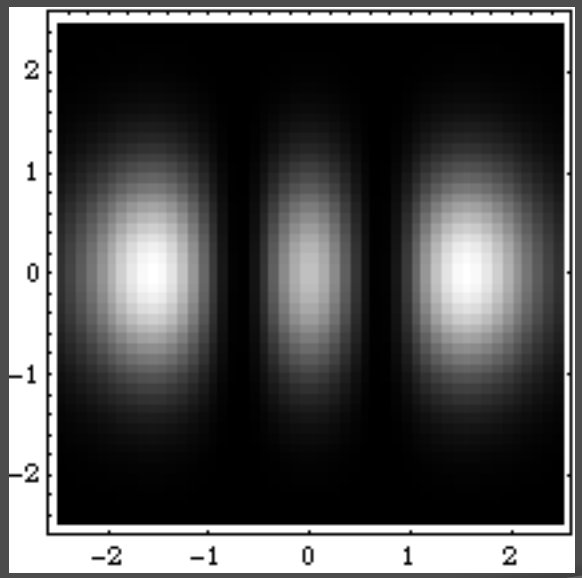
n=0

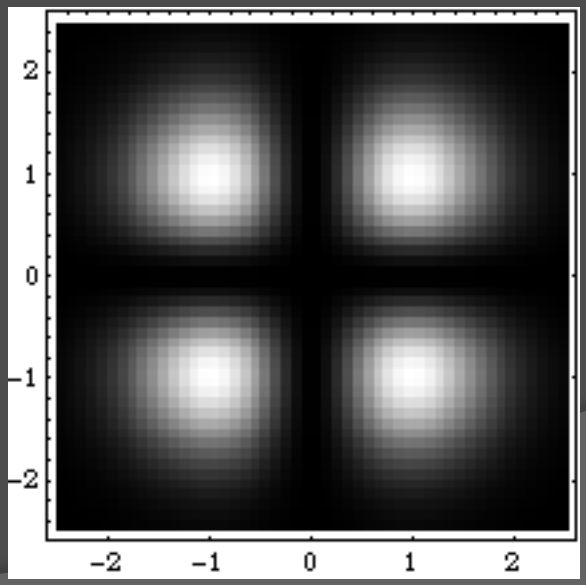




n=1

n=2
(2,0,0)

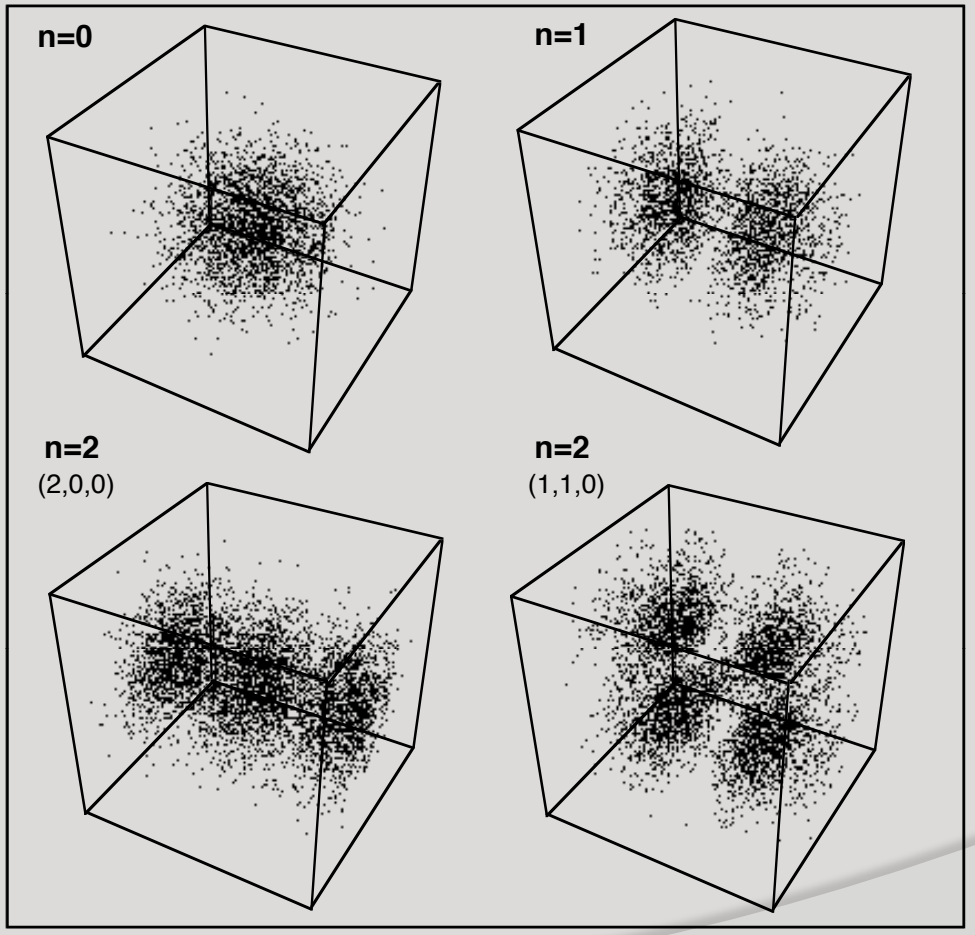




n=2
(1,1,0)

Hermite polynomials

Solutions: 3D harmonic oscillator potential



Simulation of geodesics

Kepler central potential GM/r

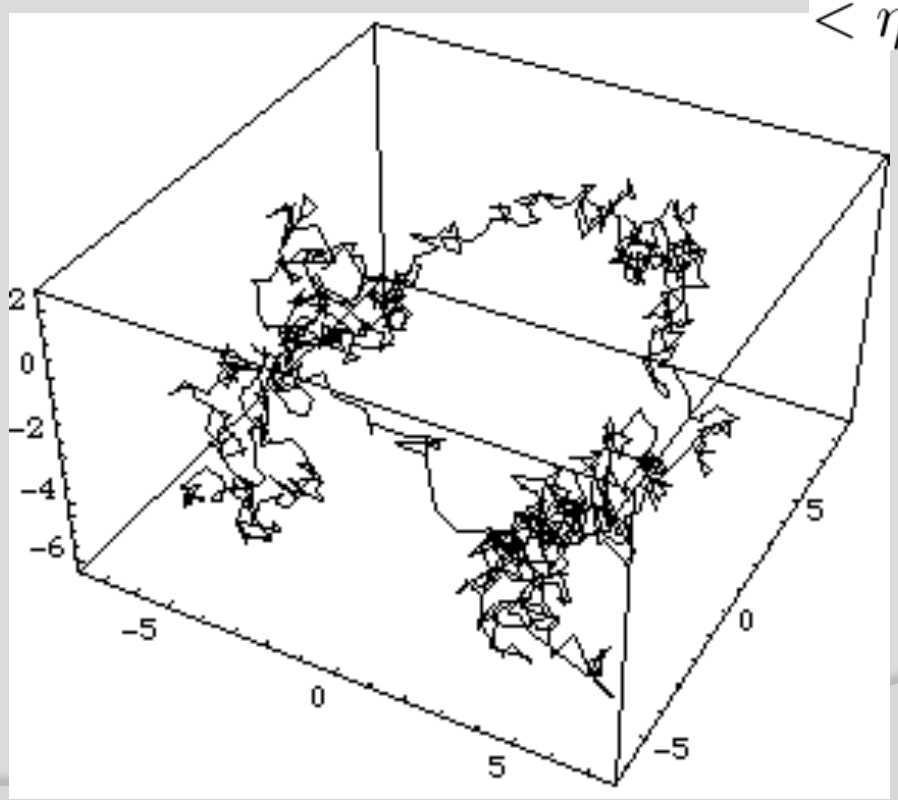
State $n = 3, l = m = n\text{-}1$

Process:

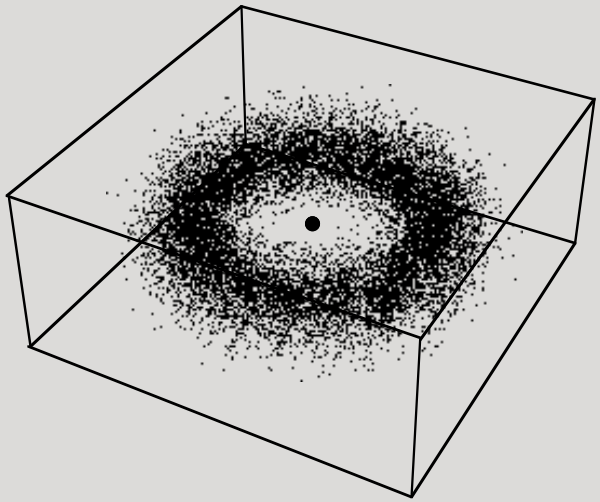
$$dx = v_+ dt + \eta\sqrt{2\mathcal{D}dt}$$

$$v_+ = U + V \qquad \mathcal{V} = V - iU$$

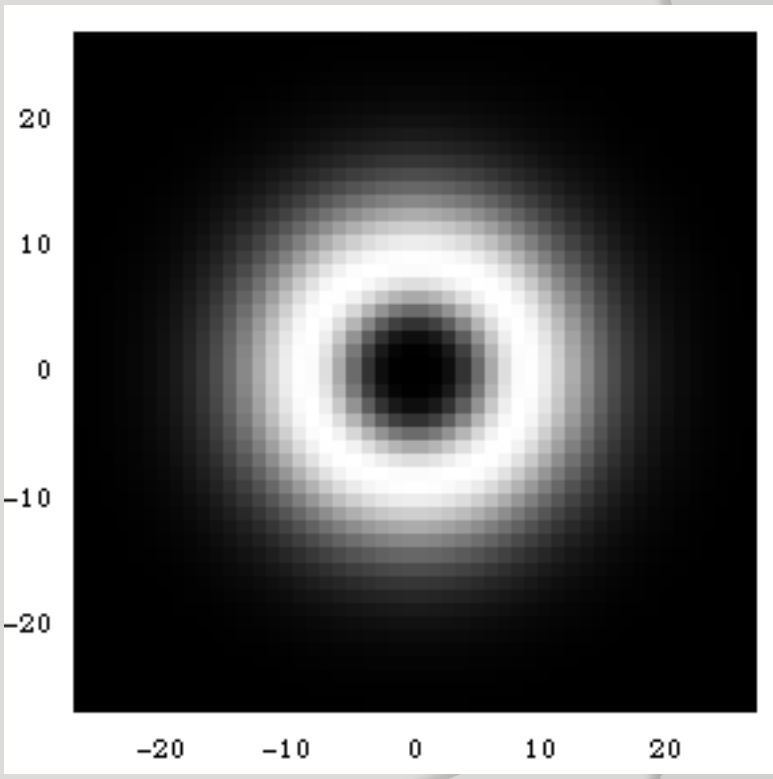
$$< \eta >= 0, \quad < \eta^2 >= 1$$



Solutions: Kepler potential

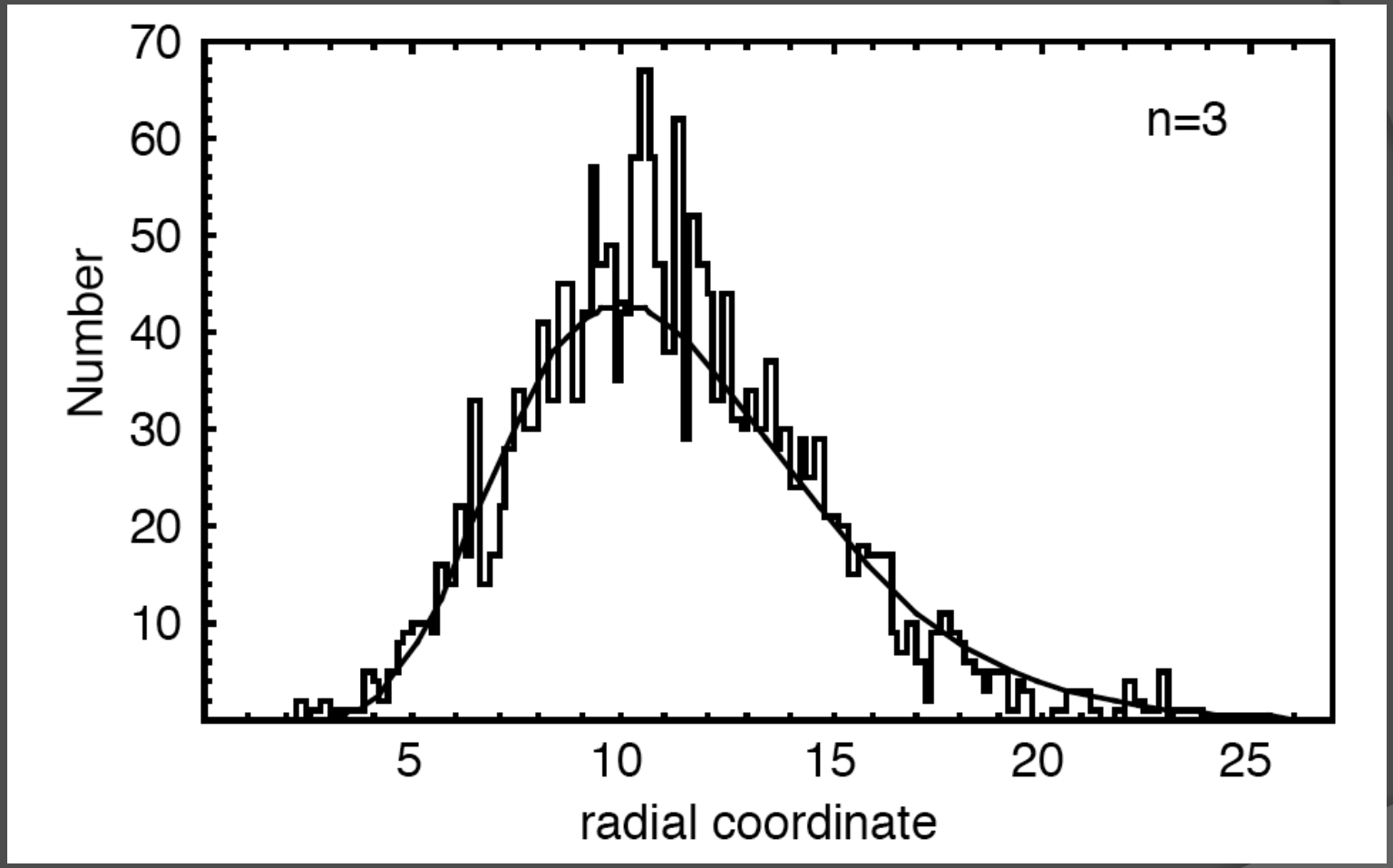


n=3



Generalized Laguerre polynomials

Hydrogen atom



Distribution obtained from one geodesical line, compared to theoretical distribution solution of Schrödinger equation