SYMBOLIC INVESTIGATION OF POLE-ZERO CANCELLATION)R A DOUBLE INVERTED PENDULUM USING TORQUE 'PUTS.

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INTRODUCTION

lern computational packages are able to perform highly complex manipulations of symbolic variables, thus 'ing insights into a variety of mathematical problems. The following investigation uses the symbolic ma- lation packages REDUCE and MATLAB Symbolic Toolbox in an investigation into the controllability with ect to torque inputs of a double inverted pendulum. The classic inverted pendulum model, and its control- ity with respect to the force applied to the trolley, have previously been investigated algebraically, following :pected results during exploratory numerical computations. Thus, the anticipated cancelling pole-zero term he system when it is in a state of uncontrollability has been explicitly identified in fully general form [1]. results presented here follow on from this investigation by considering each of the possible control inputs he model in turn, and obtaining the fully generalised form of the polynomials associated with the transfer tions for each input. The ability of the software to factorise complicated multi-variable polynomials is again oited, enabling verification of unexpected numerical results from the symbolic equations.

INVERTED PENDULUM MODEL

The System

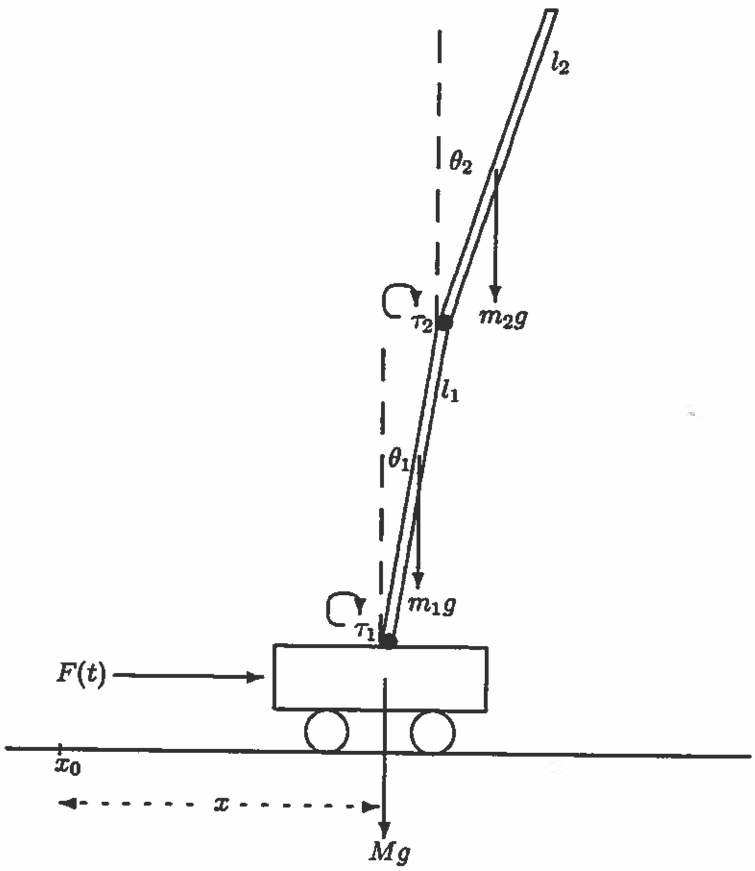

Figure 1: The damped translating double pendulum

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Consider the system of Figure 1, in which a horizontally translating inverted pendulum of two links $l^{(1)}$, $l^{(2)}$ moves under the action of a single control input, selected from the following: $F(t)$, $\tau_1(t)$ and $\tau_2(t)$. Generalised coordinates $x(t), \theta_1(t), \theta_2(t)$ are attached to the system, with x_0 some reference point on the track, and coordinates $\theta_1(t), \theta_2(t)$ measured positive clockwise from a local vertical. The trolley is assumed to have mass M (kg), with link $l^{(i)}$ of mass m_i (kg) and length l_i (m), $i = 1, 2$; the unit of time is the second (s). Acceleration due to gravity, which acts downwards in the plane, is written as g with units ms^{-2}. System motion is assumed to be damped by viscous friction throughout, with $\lambda_x < 0$ $(kg\,s^{-1})$ the coefficient of damping along the horizontal, and $\lambda_1, \lambda_2 < 0$ $(kg\,s^{-1})$ the respective coefficients of damping at the base of link $l^{(1)}, l^{(2)}$. The scalar control input acting on the pendulum will be one of the following: $u(t) = F(t)$, $u(t) = \tau_1(t)$, $u(t) = \tau_2(t)$, and the state vector $x(t)$ comprises the three generalised coordinates and their time derivatives:

$$x(t) = \begin{bmatrix} x(t) & \theta_1(t) & \theta_2(t) & \dot{x}(t) & \dot{\theta}_1(t) & \dot{\theta}_2(t) \end{bmatrix}^T \tag{1}$$

2.2 The Balancing Problems

Denote any system state of unstable equilibrium, at some arbitrary point along the track, by S_{θ_1,θ_2}, where θ_1, θ_2 define the respective alignment of each link l_1, l_2. There are clearly three such states for the double pendulum, written $S_{0,0}$, $S_{0,\pi}$, $S_{\pi,0}$ each of which gives rise to a valid balancing problem. The first state $S_{0,0}$ describes that of the standard inverted double pendulum, and is the most commonly studied, whilst the other two are non standard problems, of which the third is not strictly a balancing problem, since the first link is allowed to hang below the track level. These target states may be referred to in order as "up-up", "up-down" and "down-up", and henceforth the respective control assignments attached to them will be called Balancing Problems I, II, and III. Balancing Problems II and III are clearly mirror images of each other about the horizontal. This work concerns itself with Balancing Problem I.

3 THEORY: NON-CONTROLLABILITY AND POLE-ZERO CANCELLATION

3.1 Kalman Controllability

Formulation of the full non-linear equations of motion of the system is straightforward, and symbolic linearisation about a state of unstable equilibrium can be affected in the usual manner. The theoretical feasibility of control by $u(t)$ is given by the well-known Kalman controllability test. Each balancing problem gives rise to the linear system:

$$\frac{d(\delta \underline{x})}{dt} = [A]\,\delta\underline{x} + \underline{b}\delta G \tag{2}$$

for perturbations $\delta\underline{x}, \delta G$ in $\underline{x}, G$ about the operating point. G is the control input, which will take each of the possible forces in turn, namely F, τ_1 and τ_2. Construction of the system controllability matrix yields a necessary and sufficient condition regarding the controllability or otherwise of the pendulum with respect to the control input selected. The system is controllable if and only if the controllability matrix

$$C = [\underline{b} : [A]\,\underline{b} : [A^2]\,\underline{b} : [A^3]\,\underline{b} : [A^4]\,\underline{b} : [A^5]\,\underline{b}] \tag{3}$$

has full rank, so symbolic evaluation of the determinant of the controllability matrix will give a controllability criterion in algebraic form. Earlier work on the pendulum model has established the following criteria for the different control inputs [2,3].

Controlling the system with respect to F, the force on the trolley, gives rise to an algebraic controllability condition for each balancing problem, obtained from the symbolic evaluation of the determinant of the controllability matrix, having the general form:

$$c = c(\lambda_1, \lambda_2, m_1, m_2, l_1, l_2, g) \tag{4}$$

where the condition for non-controllabilty is $c = 0$,which is equivalent to the matrix C being singular, and hence not having full rank (see Appendix for MATLAB symbolic code and output).

Controlling the system with respect to either τ_1, the torque on the first link, or τ_2, the torque on the second link, gives rise to a controllability matrix which has a zero determinant [4,5]. The system in either of these cases is therefore always uncontrollable and hence there is no associated controllability condition. The MATLAB code for checking the determinant of the controllability matrix for Balancing Problem I, for any of the three control inputs may be found in the Appendix.

3.2 Transfer Functions

Non-controllability of the system may also be investigated by considering the transfer function. With reference to the state vector (1) the linearised output equation:

$$\delta\underline{y} = D\delta\underline{x} \tag{5}$$

which accompanies (2) is assumed, where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

so that the outputs $y_1(t)$, $y_2(t)$, $y_3(t)$ are simply the states x, θ_1, θ_2. Transfer functions relating small changes in the input to those of the outputs are given by the equation:

$$\frac{\delta\underline{y}(s)}{\delta F(s)} = D\,[sI - A]^{-1}\,\underline{b} \tag{7}$$

where I is the 6x6 identity matrix. The components of (7) may be written as:

$$\frac{\delta y_i}{\delta F(s)} = f_i(s) \quad i = 1, 2, 3 \tag{8}$$

where each of the functions f_i is a quotient of univariate polynomials in s. From standard theory, non-controllability manifests itself as pole-zero cancellation in the transfer functions, and it is this theory which forms the basis of the following work.

4 RESULTS

Recent work [1] has investigated pole-zero cancellation for the double inverted pendulum with respect to the control input F, both numerically and algebraically. Surprisingly, it was demonstrated that the cancelling pole has a linear relationship with respect to λ_1, the friction on the first link. For the quasi-uniform system, the relationship between s and λ_1 for the cancelling pole was shown to be:

$$s_c = -\frac{3}{2\epsilon_m \epsilon_l^2}\lambda_1$$

The cancelling pole with respect to the other possible control inputs has been investigated both numerically and algebraically for Balancing Problem I, and the findings are now presented.

Consider the quasi-uniform model, for which

$$M = m_1 = m_2 = \epsilon_m \quad l_1 = l_2 = \epsilon_l \tag{9}$$

In addition, let $\lambda_x = 0$. Although the damping conditions are assumed non-zero, since λ_x plays no part in the controllability condition for the system it is dispensed with for convenience. Algebraic computation implemented according to the preceding section gives a curve of non-controllability for the control input F

$$c(\lambda_1, \lambda_2, \epsilon_m, \epsilon_l, g) = \lambda_1(3\lambda_1 + 16\lambda_2) - 4g\epsilon_m^2\epsilon_l^3 = 0 \tag{10}$$

with bounds on λ_1, λ_2 given by the inequalities

$$-2\epsilon_m\epsilon_l\sqrt{\frac{g\epsilon_l}{3}} < \lambda_1 < 0, \quad -\infty < \lambda_2 < 0 \tag{11}$$

For the control inputs τ_1, τ_2 the system is always uncontrollable, so there is no condition of uncontrollability.

Consider, numerically, the case when $\lambda_1 = -1.2$ and $\lambda_2 = -1.81875$, and $\epsilon_m = \epsilon_l = 1$ so that the system is uncontrollable with regard to the control input F. Taking each of the control inputs in turn yields the following poles and zeros for each output:
with respect to F

poles:	0	0	−33.7882	−4.1464	3.4346	1.8
x zeros:	−28.3962	−2.9043	2.5005	1.8		
θ_1 zeros:	−16.350	1.800	0	0		
θ_2 zeros:	1.8	49.05	0	0		

with respect to τ_1

poles:	0	0	−33.7882	−4.1464	3.4346	1.8
x zeros:	−16.3500	-1×10^{-8}	1×10^{-8}	1.8		
θ_1 zeros:	−9.3691	2.0941	0	0		
θ_2 zeros:	7.275	0	0	0		

with respect to τ_2

poles:	0	0	−33.7882	−4.1464	3.4346	1.8
x zeros:	0	-1×10^{-8}	1×10^{-8}	1.8		
θ_1 zeros:	−3.1321	3.1321	0	0		
θ_2 zeros:	3.5434	−4.9834	0	0		

So with F as the control input, and for an uncontrollable system with respect to F, there is a cancelling pole in each transfer function of 1.8, as anticipated since for the given values of λ_1, λ_2, ϵ_m and ϵ_l :

$$\frac{3}{2\epsilon_m \epsilon_l^2} = 1.8$$

With τ_1 as control input we again have pole-zero cancellation in each transfer function, as expected from the theory. It can be seen that the cancelling pole is now zero (allowing for numerical error) and is in fact a double cancellation. In the case of the θ_2 transfer function the numerator has a triple zero at $s = 0$, so there is only one non-zero value for s. Unexpectedly, the x transfer function retains a zero at $s = 1.8$, although it is no longer a cancelling zero. With τ_2 as control input there is a double cancellation of the zero pole in each transfer function. In this case the x transfer function has a further zero at $s = 0$, and has again retained the (non-cancelling) zero at $s = 1.8$. Also of note is that the θ_1 transfer function has a pair of zeros which are of the same magnitude but opposite sign. It is of particular interest to note that with either of the torques as control input there is no uncontrollability condition, and there is apparently no dependence of the cancelling pole on any of the system parameters.

Now consider, numerically, the case when $\lambda_1 = -2$ and $\lambda_2 = -1.2$ and $\epsilon_m = \epsilon_l = 1$, so that the system is controllable with regard to the control input F. Taking each of the control inputs in turn yields the following poles and zeros for each output:
with respect to F

poles:	0	0	−26.4591	−4.3062	2.4062	3.1591
x zeros:	−21.6328	−3.0208	2.3298	2.4381		
θ_1 zeros:	−12.0436	2.4436	0	0		
θ_2 zeros:	2.2928	38.5072	0	0		

with respect to τ_1

poles:	0	0	−26.4591	−4.3062	2.4062	3.1591
x zeros:	−12.0436	-1×10^{-8}	1×10^{-8}	2.4436		
θ_1 zeros:	−7.4379	2.6379	0	0		
θ_2 zeros:	4.800	0	0	0		

with respect to τ_2

poles:	0	0	−26.4591	−4.3062	2.4062	3.1591
x zeros:	3.0000	-1×10^{-8}	$1 \times 10^{-8} + 1 \times 10^{-8}i$	$1 \times 10^{-8} - 1 \times 10^{-8}i$		
θ_1 zeros:	−3.1321	3.1321	0	0		
θ_2 zeros:	3.1701	−5.5701	0	0		

As expected, there are now no common cancelling poles in the transfer function with F as the control input. With τ_1 as control input, the expected pole-zero cancellation is evident, and again is a double cancellation of the zero poles (allowing for numerical error). The x transfer function no longer retains a third zero at $s = 0$, or a zero at $s = 1.8$. The θ_2 transfer function again only has one non-zero element.
With τ_2 as the control input, the double cancellation of the zero poles is apparent, allowing for numerical error. The x transfer function now has a third zero at $s = 0$, and the non-zero value is $s = 3.0$. Again unexpectedly the θ_1 transfer function is unaffected by the changes in the values of λ_1 and λ_2, thus retaining the two zeros which are of the same magnitude but opposite sign. Once again, with either of the torques as control input there is no uncontrollability condition, and there is no apparent dependence of the cancelling pole on any of the system parameters.

The variety of unexpected results from the numerical investigations lead us to a symbolic investigation of the transfer functions. Let us now consider the quasi-uniform system with $\lambda_x = 0$ symbolically: From REDUCE, the denominator of each transfer function is of the form s^2 into a polynomial of order 4 :

$$s^2[\epsilon_l^4\epsilon_m^2 s^4 - (3\epsilon_l^2\epsilon_m\lambda_1 + 16\epsilon_l^2\epsilon_m\lambda_2)\, s^3 + (12\lambda_1\lambda_2 - 8\epsilon_l^3\epsilon_m^2 g)\, s^2 + (6\epsilon_l\epsilon_m g\lambda_1 + 24\epsilon_l\epsilon_m g\lambda_2)\, s + 9\epsilon_l^2\epsilon_m^2 g^2]/\epsilon_l^4\epsilon_m^2 \quad (12)$$

With F as the control input, the three numerators are:

$$\begin{array}{ll} x & -[7\epsilon_l^4\epsilon_m^2 s^4 - (12\epsilon_l^2\epsilon_m\lambda_1 + 96\epsilon_l^2\epsilon_m\lambda_2)\, s^3 - (42\epsilon_l^3\epsilon_m^2 g - 36\lambda_1\lambda_2)\, s^2 + (18\epsilon_l\epsilon_m g\lambda_1 + 72\epsilon_l\epsilon_m g\lambda_2)\, s \\ & \quad +27\epsilon_l^2\epsilon_m^2 g^2]/9\epsilon_l^4\epsilon_m^3 \\ \theta_1 & s^2\left[\epsilon_l^2\epsilon_m s^2 - 8\lambda_2 s - 3\epsilon_l\epsilon_m g\right]/\epsilon_l^3\epsilon_m^2 \\ \theta_2 & -s^2\left[\epsilon_l^2\epsilon_m s^2 + (6\lambda_1 + 24\lambda_2)\, s + 9\epsilon_l\epsilon_m g\right]/3\epsilon_l^3\epsilon_m^2 \end{array} \quad (13)$$

With τ_1 as the control input the three numerators are:

$$\begin{array}{ll} x & s^2\left[\epsilon_l^2\epsilon_m s^2 - 8\lambda_2 s - 3\epsilon_l\epsilon_m g\right]/\epsilon_l^3\epsilon_m^2 \\ \theta_1 & -3s^2\left[\epsilon_l^2\epsilon_m s^2 - 4\lambda_2 s - 2\epsilon_l\epsilon_m g\right]/\epsilon_l^4\epsilon_m^2 \\ \theta_2 & 3s^3\left[\epsilon_l^2\epsilon_m s + 4\lambda_2\right]/\epsilon_l^4\epsilon_m^2 \end{array} \quad (14)$$

With τ_2 as the control input the three numerators are:

$$\begin{array}{ll} x & -s^3\left[2\epsilon_l^2\epsilon_m s + 3\lambda_1\right]/3\epsilon_l^3\epsilon_m^2 \\ \theta_1 & 6s^2\left[\epsilon_l s^2 - g\right]/\epsilon_l^3\epsilon_m \\ \theta_2 & -2s^2\left[5\epsilon_l^2\epsilon_m s^2 - 6\lambda_1 s - 9\epsilon_l\epsilon_m g\right]/\epsilon_l^4\epsilon_m^2 \end{array} \quad (15)$$

As anticipated, with F as the control input there are no cancelling poles common to all three transfer functions. With τ_1 as the control input the double cancelling pole at $s = 0$ is present in all three transfer functions, as predicted from the numerical results. The appearance of these poles algebraically is satisfying, since one might deduce that a system with no condition for uncontrollability has a cancelling pole independent of any system variables. The θ_2 numerator is in the form $s^3 p(s)$, and it can be seen that the value of s depends on λ_2, ϵ_m and ϵ_l. The x and θ_1 numerators are both in the form $s^2 p(s^2)$, and in each case the polynomial depends only on λ_2, ϵ_m and ϵ_l. With τ_2 as the control input the double cancelling pole at $s = 0$ is present in all three transfer functions, again as predicted, which is again a pleasing result. The x numerator now factorises into $s^3 p(s)$ where:

$$p(s) = 2\epsilon_l^2\epsilon_m s + 3\lambda_1$$

which is algebraically identical to the form of the cancelling pole for a system uncontrollable with respect to F. The θ_1 numerator is in the form $s^2 p(s^2)$ and since $p(s^2)$ contains only the s^2 term and a constant term, the non-zero values of s will be:

$$s = \pm\sqrt{\frac{g}{\epsilon_l}}$$

i.e. of the same magnitude and opposite sign (and independent of λ_1, λ_2 and ϵ_m), as seen numerically. The θ_2 numerator is in the form $s^2 p(s^2)$ and $p(s^2)$ depends only on λ_1, ϵ_m and ϵ_l.

In summary, for a system which is controllable with respect to F, the form of the numerators of each transfer function for each control input, ignoring any general constants, are given in the following table.

$$
\begin{array}{cccc}
 & F & \tau_1 & \tau_2 \\
x & p\left(s^4\right) & s^2 p\left(s^2\right) & s^3\left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right] \\
\theta_1 & s^2 p\left(s^2\right) & s^2 p\left(s^2\right) & s^2\left[s^2-\dfrac{g}{\epsilon_l}\right] \\
\theta_2 & s^2 p\left(s^2\right) & s^3\left[s+\dfrac{4\lambda_2}{\epsilon_l^2\epsilon_m}\right] & s^2 p\left(s^2\right)
\end{array}
\qquad (16)
$$

If the system is uncontrollable with respect to F, then $\lambda_1(3\lambda_1+16\lambda_2)=4g\epsilon_l^3\epsilon_m^2$. Substituting for λ_2 in the denominator and factorising gives:

$$
s^2\left[3\lambda_1+2\epsilon_l^2\epsilon_m s\right]\left[2\epsilon_l^2\epsilon_m\lambda_1 s^3-\left(3\lambda_1^2+8g\epsilon_l^3\epsilon_m^2\right)s^2+2g\epsilon_l\epsilon_m\lambda_1 s+12g^2\epsilon_l^2\epsilon_m^2\right]/4\lambda_1\epsilon_l^4\epsilon_m^2 \qquad (17)
$$

Thus, the denominator now factorises to the familiar form $s^2\left(3\lambda_1+2\epsilon_l^2\epsilon_m s\right)p\left(s^3\right)$. Substituting in the condition for uncontrollability with respect to F into the numerators of the transfer functions gives algebraic expressions for each numerator (after some manipulation, and ignoring general constants) as summarised in the following table.

$$
\begin{array}{cccc}
 & F & \tau_1 & \tau_2 \\
x & \left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right]p\left(s^3\right) & s^2\left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right]\left[s-\dfrac{2g\epsilon_l\epsilon_m}{\lambda_1}\right] & s^3\left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right] \\
\theta_1 & s^2\left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right]\left[s-\dfrac{2g\epsilon_l\epsilon_m}{\lambda_1}\right] & s^2 p\left(s^2\right) & s^2\left[s^2-\dfrac{g}{\epsilon_l}\right] \\
\theta_2 & s^2\left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right]\left[s+\dfrac{6g\epsilon_l\epsilon_m}{\lambda_1}\right] & s^3\left[s+\left(\dfrac{g\epsilon_l\epsilon_m}{\lambda_1}-\dfrac{3\lambda_1}{4\epsilon_l^2\epsilon_m}\right)\right] & s^2 p\left(s^2\right)
\end{array}
\qquad (18)
$$

Now, as expected, the cancelling pole in its full algebraic form is present in each transfer function when F is the control input [1]. Furthermore, the numerators for θ_1 and θ_2 are able to be fully factorised, so that each zero is available in explicit symbolic form. It is clear that these two numerators are closely related, since we may write them as follows:

$$
\begin{array}{cc}
\theta_1 & s^2\left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right]\left[s-\beta\right] \\
\theta_2 & s^2\left[s+\dfrac{3\lambda_1}{2\epsilon_l^2\epsilon_m}\right]\left[s+3\beta\right]
\end{array}
$$

where $\beta=\dfrac{2g\epsilon_l\epsilon_m}{\lambda_1}$

With τ_1 as control input, the double zero at $s=0$ is evident in each numerator. The x numerator is able to be fully factorised and thus it can be seen that one of the non-zero factors is algebraically identical to the cancelling pole present in each transfer function when F is the control input. The remaining factor is identical to that for the θ_1 numerator with F as the control input: $[s-\beta]$. The θ_2 numerator is also able to be fully factorised, and furthermore is identical to the numerator for a system controllable with respect to F since:

$$
\lambda_1\left(3\lambda_1+16\lambda_2\right)=4g\epsilon_l^3\epsilon_m^2 \Rightarrow \frac{4\lambda_2}{\epsilon_l^2\epsilon_m}=\frac{g\epsilon_l\epsilon_m}{\lambda_1}-\frac{3\lambda_1}{4\epsilon_l^2\epsilon_m}
$$

With τ_2 as control input, the double zero at $s=0$ is evident in each numerator, and in fact all three numerators are identical for both a system controllable with respect to F and a system uncontrollable with respect to F. Although the full form of $p\left(s^2\right)$ for the θ_2 numerators have not been given in the tables, in each case:

$$
p\left(s^2\right)=5\epsilon_l^2\epsilon_m s^2-6\lambda_1 s-9g\epsilon_l\epsilon_m
$$

and clearly substitution for λ_2 from the uncontrollability condition will have no effect.

Finally, the effect of the friction on the trolley has been investigated. With F as the control input, the only difference is in the denominator, which for either a controllable or uncontrollable system loses one of the zero poles, so that for example the denominator for an uncontrollable system is now of the form:

$$
s\left(3\lambda_1+2\epsilon_m\epsilon_l^2\right)p\left(s^4\right)
$$

With τ_1 as the control input the θ_1 and θ_2 numerators and the denominator lose one of the factors of s, so that there is now only a single cancellation at $s=0$ in each transfer function. The θ_2 numerator is no longer fully factorisable, and both the θ_1 and θ_2 numerators now depend on $\lambda_x, \lambda_2, \epsilon_m$ and ϵ_l. Interestingly, the x numerator is unaffected by the friction on the trolley. When τ_2 is the control input, the θ_1 and θ_2 numerators and the denominator lose one of the factors of s, and again the x numerator is unaffected by the friction on the trolley. Substituting the controllability condition into each transfer function, and comparing with the transfer functions obtained when $\lambda_x=0$ it can be seen that the x numerator is unaffected by the friction on the trolley. The denominator and the θ_1 and θ_2 numerators again only have a single factor at $s=0$,and depend on $\lambda_x, \lambda_2, \epsilon_m$ and ϵ_l.

5 SUMMARY

The continuation of the symbolic-numeric investigation previously reported [1] by considering the two torques as control inputs has given some interesting results with regard to the algebraic form of the transfer functions. Once again, unexpected numerical results have led to symbolic investigations using computer algebra, and further features of the pendulum system have been exposed. It is satisfying to find that for a control input which leads to an unconditionally uncontrollable system, the cancelling pole has no dependence on any of the system parameters.

The appearance of the cancelling pole in the x transfer functions for systems with τ_1 and τ_2 as the control inputs only when the system is uncontrollable with respect to F is an unexpected result which has been demonstrated both numerically and algebraically. Furthermore, the effect of the trolley friction on the transfer functions has been clarified and it has been shown that setting the trolley friction to zero gives an extra factor at $s=0$ in the denominator and the θ_1 and θ_2 numerators for each control input. Interestingly, the x transfer function is unaffected by the trolley friction.

The other numerical results noted have been supported by the symbolic investigation, and in some cases the transfer functions have been obtained in fully factorised form. Future work could be directed at the other two balancing problems in the first instance, with further extension to pendulums with more than two links.

6 ACKNOWLEDGEMENT

The author acknowledges the assistance of Mr. I.C. Brown who provided the various transfer functions in symbolic form using the symbolic manipulation package REDUCE.

7 APPENDIX

7.1 MATLAB Code

```
function absym
%
% ABSYM.m - program to set up the A and B matrices for the 2-link
% pendulum in symbolic form. LAM1 is the damping for link 1, LAM2
% is the damping for link 2, LAMX is the trolley damping, EM is
% the mass of each link and the trolley (assumed equal) and EL is
% the length of each link (assumed equal).
%
% usage: absym
%
syms lam1 lam2 el em adag astar a bfull real;
syms c1 c2 c3 c4 c5 real;
g = 9.81;
lamx = 0;
%
% Setting up adag
%
adag(1,1) = 0;
adag(2,1) = 0;
adag(3,1) = 0;
adag(1,2) = -3*g/2;
adag(2,2) = 9*g/(2*el);
adag(3,2) = -9*g/(2*el);
adag(1,3) = g/6;
adag(2,3) = -3*g/(2*el);
adag(3,3) = 7*g/(2*el);
%
% Setting up astar
%
astar(1,1) = 7*lamx/(9*em);
astar(2,1) = -lamx/(em*el);
astar(3,1) = lamx/(3*em*el);
astar(1,2) = -(3*lam1 + 4*lam2)/(3*em*el);
astar(2,2) = 3*(lam1 + 2*lam2)/(em*el*el);
astar(3,2) = -(3*lam1 + 10*lam2)/(em*el*el);
astar(1,3) = 4*lam2/(3*em*el);
astar(2,3) = -6*lam2/(em*el*el);
astar(3,3) = 10*lam2/(em*el*el);
%
% Setting up bdag
%
bdag(1,1) = 7/(9*em);
bdag(2,1) = -1/(em*el);
bdag(3,1) = 1/(3*em*el);
bdag(1,2) = -1/(el*em);
bdag(2,2) = 3/(em*el*el);
bdag(3,2) = -3/(em*el*el);
bdag(1,3) = 4/(3*em*el);
bdag(2,3) = -6/(em*el*el);
bdag(3,3) = 10/(em*el*el);
%
% Setting up a and b
%
a(1:3,1:3) = zeros(3);
a(1:3,4:6) = eye(3);
```

```
a(4:6,1:3) = adag;
a(4:6,4:6) = astar;
bfull(1:3,1:3) = zeros(3);
bfull(4:6,1:3) = bdag;
% Selecting control input
fprintf('F 1\n');
fprintf('Tau1 2\n');
fprintf('Tau2 3\n');
ci = input('Please select number for control input: ');
% Setting up b matrix for given control
if (ci == 1)
 b = bfull(:,1);
 elseif (ci == 2)
 b = bfull(:,2);
 else
 b = bfull(:,3);
 end

c1 = a*b;
c2 = a*c1;
c3 = a*c2;
c4 = a*c3;
c5 = a*c4;
cmat = [b c1 c2 c3 c4 c5];
ceq1 = det(cmat);
ceq = factor(ceq1);
fprintf(['The characteristic equation is ' char(ceq) '\n']);
if (ceq ~= 0)
 solve(ceq,'lam1')
 end
```

7.2 Output

Using F as the control input:

```
>> absym
F 1
Tau1 2
Tau2 3
Please select number for control input: 1
The characteristic equation is 8176878532160109/62500000000/el^13/em^8*(-75*lam1^2+
981*em^2*el^3-400*lam1*lam2)

ans =

[ -8/3*lam2+1/15*(1600*lam2^2+2943*em^2*el^3)^(1/2)]
[ -8/3*lam2-1/15*(1600*lam2^2+2943*em^2*el^3)^(1/2)]
```

Using τ_1 as the control input:

```
>> absym
F 1
Tau1 2
Tau2 3
Please select number for control input: 2
The characteristic equation is 0
```


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