ON A VECTOR Q-D ALGORITHM

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Abstract

Using the framework provided by Clifford algebras, we consider a non-commutative quotient-difference algorithm for obtaining the elements of a continued fraction corresponding to a given vector-valued power series. We demonstrate that these elements are ratios of vectors, which may be calculated with the aid of a *cross rule* using *only* vector operations. For vector-valued meromorphic functions we derive the asymptotic behaviour of these vectors, and hence of the continued fraction elements themselves. The behaviour of these elements is similar to that in the scalar case, while the vectors are linked with the residues of the given function. In the particular case of vector power series arising from matrix iteration the new algorithm amounts to a generalisation of the power method to sub-dominant eigenvalues, and their eigenvectors.

Key words: Vector continued fraction, vector Padé approximant, quotient-difference algorithm, Clifford algebra, cross rule, power method.

1 Introduction

The theory of vector Padé approximants is concerned with rational approximations to vector-valued functions given in the form of power series. This theory may be developed, with the aid of Clifford algebras [17], in a manner which follows that of the well-established theory for rational approximants of real or complex-valued functions [1,3]. In particular, if a *corresponding* continued fraction is known, then three-term recurrence relations may be used to construct the numerator and denominator polynomials involved. In this context, vector versions of the Viskovatov and Modified

Euclidean algorithms have been established [6] which allow the determination of the elements of certain types of continued fraction corresponding to a given vector-valued function.

In this paper we establish a cross rule, based on the non-commutative quotient-difference algorithm, for calculating the elements of an *equivalent* continued fraction whose partial denominators are unity. These elements are shown to be ratios of vectors, which we label by $\mathbf{U}_m^J$. The cross rule gives rise to a new algorithm which may be implemented using *vectors only*, thus avoiding general Clifford numbers. In section 5 we discuss the construction of the *full* q-d and $\mathbf{U}$-tables, thus furnishing the elements involved in continued fraction representations of *any* vector Padé approximant. We are then able to construct the polynomials of a given approximant. This complements the vector ϵ-algorithm, which is normally used to produce *values* of vector Padé approximants.

For reasons of clarity, the initial presentation assumes that the power series coefficients are *real* vectors. There is more than one possible extension to *complex* vectors, each corresponding to a different definition of the inverse of a vector. This topic is discussed briefly in section 6.

We then consider power series for vector-valued meromorphic functions and derive results for the asymptotic behaviour of the columns of the $\mathbf{U}$-table. As a consequence of this, it is discovered that the columns of the q-d table behave in a manner similar to those for the scalar case *e.g.*[10]. In the course of proving these points we demonstrate the connection between the entries in the $\mathbf{U}$-table and the vector-valued residues of the given function. Furthermore, the above results determine the asymptotic behaviour of the vectors involved in the Viskovatov algorithm mentioned earlier.

Finally, we indicate some aspects of the application of the cross rule to vector sequences produced by matrix iteration. In particular we note that, for the $\mathbf{U}$-table associated with continued fractions corresponding to the vector-valued function generating the iterates, the column entries tend to eigenvectors of the iteration matrix. In the case of the eigenvalues having distinct moduli, this amounts to a generalisation of the power method to compute *all* the eigenvalues and their eigenvectors, given *only* the initial power iterates. A simple example illustrating these points is given.

2 Some Notation

We let $C\ell_d$ denote the real Clifford algebra of $\mathbb{R}^d$ [12,15,16]. This is the associative algebra over $\mathbb{R}$ generated by the orthonormal basis of $\mathbb{R}^d$, $\{\mathbf{e}_1, \mathbf{e}_2 \cdots \mathbf{e}_d\}$, which satisfies the anti-commutation relations

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_i\mathbf{e}_j = 2\delta_{i,j} \qquad\qquad i, j = 1, 2 \cdots, d \tag{2.1}$$

where the algebra identity is 1. We also require the universality property that $\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_d \neq \pm 1$. $C\ell_d$ is a linear space of dimension 2^d spanned by the basis elements

$$\mathbf{e}_I = \mathbf{e}_{i_1 i_2 \cdots i_k} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k} \tag{2.2}$$

where $I = \{i_1, i_2, \cdots, i_k\}$ and $1 \leq i_1 < i_2 < \cdots i_k \leq d$ for $k = 1, 2\cdots, d$. The identity element corresponds to the empty set ($k = 0$). A general element of $C\ell_d$ is given by

$$a = \sum_I a_I \mathbf{e}_I \qquad a_I \in \mathbb{R} \tag{2.3}$$

where the summation is over the 2^d different ordered multi-indices I. The coefficient a_0 is called the real or scalar part of a, and is denoted by $Re(a)$. The spinor norm or absolute value of an element is defined by

$$|a| = \sqrt{\sum_I |a_I|^2}. \tag{2.4}$$

From [12] we have

$$|ab| \leq K_d |a||b| \qquad\qquad \forall\, a, b \in C\ell_d \tag{2.5}$$

where K_d is a real positive constant whose value depends on the Clifford algebra concerned.

We shall require three involutions of $C\ell_d$. The first of which, called the *main involution*, is the isomorphism : $a \mapsto \hat{a}$ in which each $\mathbf{e}_i$ is replaced by $-\mathbf{e}_i$; hence $\widehat{ab} = \hat{a}\hat{b}$. The second one, called *reversion*, is the anti-isomorphism : $a \mapsto \tilde{a}$ obtained by reversing the order of factors in $\mathbf{e}_I$; hence $\widetilde{ab} = \tilde{b}\tilde{a}$. Finally, we combine the first two operations to form the anti-isomorphism, *conjugation* : $a \mapsto \bar{a}$ where $\bar{a} := \hat{\tilde{a}}$; hence $\overline{ab} = \bar{b}\bar{a}$.

Each vector $(v_1, v_2, \cdots, v_d) \in \mathbb{R}^d$ will be identified with an element, $\sum_{i=1}^d v_i \mathbf{e}_i$, of $C\ell_d$, using the common label $\mathbf{v}$. We use the Euclidean norm in $\mathbb{R}^d$ which is consistent with the spinor norm applied to vectors. The anti-commutation relations, (2.1), imply

$$\mathbf{uv} + \mathbf{vu} = 2(\mathbf{u}\cdot\mathbf{v}) \tag{2.6}$$

where $\mathbf{u}\cdot\mathbf{v}$ indicates the usual scalar product, $\sum_{i=1}^d u_i v_i$, and

$$\mathbf{uvu} = 2(\mathbf{u}\cdot\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{v} \tag{2.7}$$

i.e. $\mathbf{uvu} \in \mathbb{R}^d$. Using (2.6) we obtain the identity

$$\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v} \tag{2.8}$$

where $\mathbf{u}\wedge\mathbf{v}$ denotes the *bivector*

$$\sum_{i<j} (u_i v_j - v_i u_j)\mathbf{e}_{ij}.$$

The set of products of non-null vectors forms a group under multiplication — the *Lipschitz group*, Γ_d [15]. If $a \in \Gamma_d$ then $a\tilde{a} = \tilde{a}a = |a|^2$. Hence,

$$a^{-1} = \frac{\tilde{a}}{|a|^2}. \tag{2.9}$$

In particular, using (2.6),

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \qquad\qquad \mathbf{v} \in \mathbb{R}^d. \tag{2.10}$$

This is identical to the Moore-Penrose generalised inverse of a real vector see *e.g.* [5]. It may also be shown that, in contrast to (2.5), [15]

$$|ab| = |a||b| \qquad\qquad \forall\ a, b \in \Gamma_d. \tag{2.11}$$

From (2.7) and (2.10) we may deduce that the reflection of $\mathbf{v}$ in the hyperplane orthogonal to $\mathbf{u}$ is given by $\mathbf{u}\mathbf{v}\hat{\mathbf{u}}^{-1}$. Since an isometry of $\mathbb{R}^d$ may be accomplished by a sequence of reflections *c.f.* [15,16] a rotation of a vector $\mathbf{v}$ may be represented by

$$a\mathbf{v}\hat{a}^{-1} \qquad\qquad \text{for some} \quad a \in \Gamma_d. \tag{2.12}$$

Finally, we note that, since $C\ell_d$ is not a division algebra for $d > 0$, one of our tasks will be to establish sufficient conditions for the existence of those inverses required to implement the q-d algorithm.

3 Corresponding Continued Fractions

We consider a vector-valued function $\mathbf{f}(z)$ with a Maclaurin series expansion

$$\mathbf{f}(z) = \mathbf{c}_0 + z\mathbf{c}_1 + z^2\mathbf{c}_2 + \ldots, \qquad z \in \mathbb{C}, \quad \mathbf{c}_i \in \mathbb{R}^d, \quad i = 0, 1, \ldots \tag{3.1}$$

valid in some neighbourhood of the origin. The right-handed $[l/m]$ vector Padé approximant to $\mathbf{f}(z)$, if it exists, is defined by

$$[l/m](z) := p^{[l/m]}(z)[q^{[l/m]}(z)]^{-1} \tag{3.2}$$

for which

$$\mathbf{f}(z) - [l/m](z) = O(z^{l+m+1}) \tag{3.3}$$

and

$$q^{[l/m]}(0) = 1 \tag{3.4}$$

where $p^{[l/m]}(z)$ and $q^{[l/m]}(z)$ are polynomials in $z \in \mathbb{C}$ over $C\ell_d$ of maximum degrees l and m respectively [17]. The left-handed vector Padé approximant is obtained using reversion. When these approximants exist they are identical, so guaranteeing uniqueness. In addition, we have the *duality* property, which states that, if $\mathbf{f}(0) \neq \mathbf{0}$ then, using an obvious notation, in which $\mathbf{g}(z) := [\mathbf{f}(z)]^{-1}$,

$$[l/m]_{\mathbf{f}}(z) \equiv \{[m/l]_{\mathbf{g}}(z)\}^{-1}$$

provided either approximant exists [20] .

As in the scalar case these constructs may be arrayed, as shown in Fig.1, in a two-dimensional table, staircase sequences of which may be built using vector continued fractions.

$$\begin{array}{cccc}
[0/0] & [1/0] & [2/0] & \dots \\
[0/1] & [1/1] & [2/1] & \dots \\
[0/2] & [1/2] & [2/2] & \dots \\
\vdots & \vdots & \vdots &
\end{array}$$

Figure 1: Part of the vector Padé table

In [6] it is demonstrated how Viskovatov's algorithm may be adapted to determine the elements of a continued fraction, which takes the form

$$\boldsymbol{\pi}_0^J(z) + z^J[\boldsymbol{\pi}_1^J + z[\boldsymbol{\pi}_2^J + \cdots z[\boldsymbol{\pi}_k^J \cdots]^{-1} \cdots]^{-1}]^{-1} \quad \boldsymbol{\pi}_i^J \in \mathbb{R}^d, \quad i = 1, 2 \cdots \tag{3.5a}$$

in non-degenerate cases where $\boldsymbol{\pi}_i^J \neq \mathbf{0}$, $i = 1, 2 \cdots$, and

$$\boldsymbol{\pi}_0^J(z) := \sum_{i=0}^{J-1} \mathbf{c}_i z^i \quad \text{for} \quad J > 0, \quad \boldsymbol{\pi}_0^J(z) := \mathbf{0} \quad \text{for} \quad J = 0. \tag{3.5b}$$

This continued fraction *corresponds* to $\mathbf{f}(z)$; that is to say

$$\mathbf{C}_n^J(z) = \mathbf{f}(z) + O(z^{n+J}) \qquad n = 0, 1, 2 \cdots \tag{3.6}$$

where $\mathbf{C}_n^J(z)$ is the n^{th} convergent of (3.5)

$$\mathbf{C}_n^J(z) = \boldsymbol{\pi}_0^J(z) + z^J[\boldsymbol{\pi}_1^J + z[\boldsymbol{\pi}_2^J + \cdots + z[\boldsymbol{\pi}_n^J]^{-1} \cdots]^{-1}]^{-1} \tag{3.7}$$

If $n = 0$, then $\mathbf{C}_0^J(z) := \boldsymbol{\pi}_0^J(z)$. A description of how the constant vectors $\boldsymbol{\pi}_i^J$ may be calculated using an algorithm involving scalars and vectors *only* may be found in [20].

As in [6] we may write

$$\mathbf{C}_n^J(z) = p_n^J(z)[q_n^J(z)]^{-1} \tag{3.8}$$

where $p_n^J(z), q_n^J(z)$ are polynomials in $C\ell_d[z]$, of degrees $J+[n-1/2]$ and $[n/2]$, respectively, for $n \geq 1$. These polynomials satisfy the three-term recurrence relations

$$\left.\begin{array}{lll} p_n^J(z) = p_{n-1}^J(z)\boldsymbol{\pi}_n^J + zp_{n-2}^J(z) & p_{-1}^J(z) := z^{J-1}, & p_0^J(z) := \boldsymbol{\pi}_0^J(z) \\ q_n^J(z) = q_{n-1}^J(z)\boldsymbol{\pi}_n^J + zq_{n-2}^J(z) & q_{-1}^J(z) := 0, & q_0^J(z) := 1 \end{array}\right\} \tag{3.9}$$

for $n = 1, 2 \cdots$. It then follows that $q_n^J(0) = \boldsymbol{\pi}_1^J \boldsymbol{\pi}_2^J \cdots \boldsymbol{\pi}_n^J \in \Gamma_d$ and is, therefore, invertible — allowing the Baker condition (3.4) to be met. We have

$$\mathbf{C}_{2m}^J(z) \equiv [J+m-1/m](z) \quad , \quad \mathbf{C}_{2m+1}^J(z) \equiv [J+m/m](z) \tag{3.10}$$

It may be shown, using the methods of [17], that

$$q_n^J(z)\widetilde{q_n^J(z)} \in \mathbb{R}[z]. \tag{3.11a}$$

Hence, noting that $p_n^J(z)\widetilde{q_n^J(z)}$ is the polynomial given by the first $n+1$ terms of the Maclaurin expansion of $[\mathbf{f}(z)q_n^J(z)\widetilde{q_n^J(z)}]$ which, using (3.11a), has vectors for its coefficients, we obtain

$$p_n^J(z)\widetilde{q_n^J(z)} \in \mathbb{R}^d[z]. \tag{3.11b}$$

From (3.11a,b) we conclude that

$$p_n^J(z)\widetilde{p_n^J(z)} \in \mathbb{R}[z]. \tag{3.11c}$$

We label the vector polynomial in (3.11b) by $\mathbf{P}_n^J(z)$, which has maximum degree $J+n-1$, and represent the scalar polynomial $q_n^J(z)\widetilde{q_n^J(z)}$ by $Q_n^J(z)$, of degree $2[n/2]$ (where $[\xi]$ denotes the integer part of ξ). It then follows that

$$\mathbf{C}_n^J(z) = \frac{\mathbf{P}_n^J(z)}{Q_n^J(z)} \tag{3.12}$$

which is in the form of a generalised inverse Padé approximant, first defined and studied by Graves-Morris *e.g.* [5].

Furthermore, from [18],

$$p_n^J(x), q_n^J(x) \in \Gamma_d \quad \text{for each } x \in \mathbb{R}. \tag{3.13}$$

In order to recast the above continued fraction into a form appropriate for the q-d algorithm, we consider equivalence transformations. The continued fraction

$$b_0(z) + z^J a_1[b_1 + za_2[b_2 + \cdots]^{-1}]^{-1}$$

is *equivalent* to one in which the elements have undergone the *equivalence transformation* :

$$\left.\begin{array}{l} b_0'(z) = b_0(z) \;\;,\;\; a_1' = a_1\alpha_1 \;\;,\;\; b_1' = b_1\alpha_1 \;\;,\;\; a_2' = a_2\alpha_2 \\ b_i' = (\alpha_{i-1})^{-1}b_i\alpha_i \quad i \geq 2 \quad \text{and} \quad a_i' = (\alpha_{i-2})^{-1}a_i\alpha_i \quad i \geq 3 \end{array}\right\} \tag{3.14}$$

where each α_i , $i = 1, 2, \cdots$ is an invertible element of $C\ell_d$. It then follows, using an obvious notation, that

$$p_n'^J(z) = p_n^J(z)\alpha_n \quad \text{and} \quad q_n'^J(z) = q_n^J(z)\alpha_n \tag{3.15}$$

thus ensuring that the n^{th} convergent of the transformed fraction is identical to $\mathbf{C}_n^J(z)$ of eqn.(3.8). If $\alpha_n\widetilde{\alpha_n} \in \mathbb{R}$, then statements (3.11) hold for the transformed polynomials.

More generally, suppose the equivalence transformation (3.14) is performed on a continued fraction whose n^{th} convergent has numerator and denominator polynomials $p_n(z)$ and $q_n(z)$, respectively. If $\alpha_n \in \Gamma_d$ for $n = 1, 2, \cdots$, then, denoting the transformed polynomials by a prime, we may readily prove

Theorem 3.1 *(i) For each* $x \in \mathbb{R}$, $p_n'(x), q_n'(x) \in \Gamma_d \Longleftrightarrow p_n(x), q_n(x) \in \Gamma_d$.
(ii) The statements in 3.11 hold for $p_n'(z), q_n'(z)$ *if and only if they hold for* $p_n(z), q_n(z)$.

We now set

$$b_0'(z) := \sum_{k=0}^{J-1} \mathbf{c}_k z^k \quad , \quad a_i' := 1 \quad , \quad b_i' := \boldsymbol{\pi}_i^J \quad i = 1, 2, \cdots$$

and seek α_i such that $b_i = 1$ for $i = 1, 2, \cdots$. Relabelling the α_i to indicate dependence on J, we obtain

$$\alpha_i^J = \boldsymbol{\pi}_1^J \boldsymbol{\pi}_2^J \cdots \boldsymbol{\pi}_i^J \quad \in \Gamma_d \qquad i = 1, 2, \cdots. \tag{3.16}$$

Then

$$\left.\begin{array}{c} a_1 = [\boldsymbol{\pi}_1^J]^{-1} = \mathbf{c}_J \ , \ a_2 = [\boldsymbol{\pi}_2^J]^{-1}[\boldsymbol{\pi}_1^J]^{-1} \\ a_i = \mathbf{u}_i^{-1}\mathbf{v}_i^{-1} \qquad i = 3, 4, \cdots \end{array}\right\} \tag{3.17}$$

where we introduce the vectors

$$\mathbf{u}_i := R_{i-2}(\boldsymbol{\pi}_i^J) \quad \text{and} \quad \mathbf{v}_i := R_{i-2}(\boldsymbol{\pi}_{i-1}^J)$$

in which $R_i(\mathbf{w})$ denotes the rotation of $\mathbf{w}$ in $\mathbb{R}^d$ defined by

$$R_i(\mathbf{w}) := \alpha_i^J \mathbf{w} \widehat{\alpha_i^J}^{-1} \qquad i = 1, 2, \cdots \tag{3.18}$$

We note that

$$R_i(\mathbf{w}^{-1}) = [R_i(\mathbf{w})]^{-1}.$$

For even values of i, the rotation is proper, while for odd i, it is improper [15,16]. Hence, for $i > 1$ each a_i is the product of two vectors, that is to say the sum of a scalar and a bivector – see the identity (2.8). The scalar part is given by

$$Re(a_i{}^{-1}) = \boldsymbol{\pi}_i^J \cdot \boldsymbol{\pi}_{i-1}^J \quad \text{or} \quad Re(a_i) = [\boldsymbol{\pi}_i^J]^{-1} \cdot [\boldsymbol{\pi}_{i-1}^J]^{-1} \qquad i = 2, 3, \cdots \tag{3.19}$$

While the above establishes the Clifford nature of the a_i, the assumption of non-degeneracy ensures that they are invertible.

In the next section, we present an alternative method of calculating the vectors forming each a_i, based on the non-commutative q-d algorithm.

4 The q-d algorithm and a cross rule

Under the equivalence transformation defined by (3.14) and (3.16), the corresponding continued fraction (3.5) takes the following form

$$\sum_{i=0}^{J-1} \mathbf{c}_i z^i + z^J a_1[1 + za_2[1 + za_3[1 + \cdots]^{-1}]^{-1}]^{-1} \tag{4.1}$$

of which the n^{th} numerator $A_n^J(z)$, and denominator $B_n^J(z)$, satisfy the recurrence relations

$$\left.\begin{array}{l} A_n^J(z) := A_{n-1}^J(z) + zA_{n-2}^J(z)a_n \\ B_n^J(z) := B_{n-1}^J(z) + zB_{n-2}^J(z)a_n \end{array}\right\} \tag{4.2}$$

for $n = 1, 2 \cdots$, with the initial conditions

$$\left.\begin{array}{ll} A_{-1}^J(z) := z^{J-1}, & A_0^J(z) := \sum_{i=0}^{J-1} \mathbf{c}_i z^i \\ B_{-1}^J(z) := 0, & B_0^J(z) := 1 \end{array}\right\} \tag{4.3}$$

Theorem 3.1 implies that the polynomials $A_n^J(z)$ and $B_n^J(z)$ satisfy similar statements to (3.11) and (3.12), and that, for real values of z, they belong to the Lipschitz group.

From (3.6) we may write

$$\mathbf{f}(z) - A_n^J(z)[B_n^J(z)]^{-1} = \mathbf{s}_n^J z^{J+n} + O(z^{J+n+1}) \quad n = 0, 1, 2 \cdots \tag{4.4}$$

where $\mathbf{s}_n^J \in \mathrm{I\!R}^d$, which follows from the vector nature of (3.12).
Theorem 4.1

$$a_n = -[\mathbf{s}_{n-2}^J]^{-1}\mathbf{s}_{n-1}^J \quad n = 2, 3, 4 \cdots \tag{4.5}$$

with

$$a_1 = \mathbf{s}_0^J = \mathbf{c}_J \quad , \quad \mathbf{s}_1^J = \mathbf{s}_0^{J+1} = \mathbf{c}_{J+1} \tag{4.6}$$

Proof. We follow the method outlined in [11], which uses ideas similar to those in the derivation of the Berlekamp-Massey algorithm [2,13] as presented, for example, in [1]. Since $B_n^J(0) = 1, \forall n \geq 0$, (4.4) implies

$$\mathbf{f}(z)B_n^J(z) - A_n^J(z) = \mathbf{s}_n^J z^{J+n} + O(z^{J+n+1}) \quad n = 0, 1, 2 \cdots . \tag{4.7}$$

If we apply (4.2) to this order condition, we obtain, by considering the coefficient of z^{J+n-1},

$$\mathbf{s}_{n-1}^J + \mathbf{s}_{n-2}^J a_n = 0.$$

thus establishing (4.5) for $n \geq 2$. For $n = 0$ we use (4.3) to show that the coefficient of z^J in (4.4) yields $\mathbf{s}_0^J = \mathbf{c}_J$, while for $n = 1$, we obtain $\mathbf{s}_1^J = \mathbf{c}_{J+1} = \mathbf{s}_0^{J+1}$, since $A_1^J(z) = \sum_{i=0}^{J} \mathbf{c}_i z^i$ and $B_1^J(z) = 1$. □

The continued fraction (4.1) may be represented in the more familiar form

$$\sum_{i=0}^{J-1} \mathbf{c}_i z^i + z^J \mathbf{c}_J [1 - z q_1^J [1 - z e_1^J [1 - z q_2^J [1 - z e_2^J [1 - \cdots]^{-1}]^{-1}]^{-1}]^{-1}]^{-1} \tag{4.8}$$

with

$$q_m^J := -a_{2m} \quad \text{and} \quad e_m^J := -a_{2m+1} \quad m = 1, 2, \cdots \tag{4.9}$$

To derive the non-commutative q-d algorithm, we construct the even and odd parts of (4.8). This may be accomplished using a method proposed by Wynn [23] , or by adapting a scalar identity from [1] for the non-commuting case — *viz*

$$1 + z a_n [1 + z a_{n+1} D^{-1}] \equiv 1 + z a_n - z^2 a_n a_{n+1} [z a_{n+1} + D]^{-1} \tag{4.10}$$

Using this identity, we obtain for the odd part of (4.8)

$$\sum_{i=0}^{J} \mathbf{c}_i z^i + z^{J+1} \mathbf{c}_J q_1^J [1 - z(q_1^J + e_1^J) - z^2 e_1^J q_2^J [1 - z(q_2^J + e_2^J) - z^2 e_2^J q_3^J [\cdots]^{-1}]^{-1}]^{-1}$$

and for the even part, with $J \to J + 1$,

$$\sum_{i=0}^{J} \mathbf{c}_i z^i + z^{J+1} \mathbf{c}_{J+1} [1 - z q_1^{J+1} - z^2 q_1^{J+1} e_1^{J+1} [1 - z(e_1^{J+1} + q_2^{J+1}) - z^2 q_2^{J+1} e_2^{J+1} [\cdots]^{-1}]^{-1}]^{-1}$$

On identifying these expressions we establish the non-commutative quotient-difference algorithm given by :

Theorem 4.2

$$\left.\begin{array}{r} e_m^J + q_m^J = q_m^{J+1} + e_{m-1}^{J+1} \\ e_m^J q_{m+1}^J = q_m^{J+1} e_m^{J+1} \end{array}\right\} \tag{4.11}$$

for $m = 1, 2, \cdots$ *and* $J = 0, 1, 2 \cdots$, *with*

$$\left.\begin{array}{rl} e_0^J = 0 & for \;\; J = 1, 2 \cdots \\ q_1^J = [\mathbf{c}_J]^{-1} \mathbf{c}_{J+1} & for \;\; J = 0, 1, 2 \cdots \end{array}\right\} \tag{4.12}$$

as initial conditions.

We now note that, since

$$A_{2m}^{J+1}(z)[B_{2m}^{J+1}(z)]^{-1} \equiv [J + m/m](z) \equiv A_{2m+1}^J(z)[B_{2m+1}^J(z)]^{-1}$$

then

$$\mathbf{s}_{2m}^{J+1} = \mathbf{s}_{2m+1}^J.$$

Hence, on defining

$$\mathbf{U}_m^J := \mathbf{s}_{2m}^J \tag{4.13}$$

we may use (4.9) and (4.5) to express q_m^J and e_m^J as follows :

$$q_m^J = [\mathbf{U}_{m-1}^J]^{-1}\mathbf{U}_{m-1}^{J+1} \qquad e_m^J = [\mathbf{U}_{m-1}^{J+1}]^{-1}\mathbf{U}_m^J \tag{4.14}$$

for $J = 0, 1, 2\cdots$ and $m = 1, 2\cdots$.
Conversely, we have (*c.f.* [11] p527)

$$\mathbf{U}_m^J = \mathbf{c}_J q_1^J e_1^J \cdots q_m^J e_m^J. \tag{4.15}$$

The following theorem constitutes the cross algorithm.
Theorem 4.3 *The vectors* $\mathbf{U}_m^J$, *satisfy the (five point) identity*

$$\mathbf{U}_{m+1}^J = \mathbf{U}_m^{J+2} + \mathbf{U}_m^{J+1}[(\mathbf{U}_{m-1}^{J+2})^{-1} - (\mathbf{U}_m^J)^{-1}]\mathbf{U}_m^{J+1} \tag{4.16}$$

with the initialisations

$$\mathbf{U}_{-1}^J := \infty\ , \quad J = 2, 3\cdots \quad \textit{and} \quad \mathbf{U}_0^J := \mathbf{c}_J\ , \quad J = 0, 1, 2\cdots \tag{4.17}$$

Proof. Using equations (4.14), we obtain

$$e_{m+1}^J + q_{m+1}^J = [\mathbf{U}_m^{J+1}]^{-1}[\mathbf{U}_{m+1}^J + \mathbf{U}_m^{J+1}(\mathbf{U}_m^J)^{-1}\mathbf{U}_m^{J+1}]$$

and

$$e_m^{J+1} + q_{m+1}^{J+1} = [\mathbf{U}_m^{J+1}]^{-1}[\mathbf{U}_m^{J+1}(\mathbf{U}_{m-1}^{J+2})^{-1}\mathbf{U}_m^{J+1} + \mathbf{U}_m^{J+2}].$$

On multiplying these two equations from the left by $\mathbf{U}_m^{J+1}$, (4.16) follows. The other part of (4.11) is a consequence of (4.14), while the condition for $\mathbf{U}_0^J$ follows from (4.6). Finally, if we set $\mathbf{U}_{-1}^J := \infty$, $J = 2, 3\cdots$, then the initialisation for e_0^J in (4.12) is guaranteed. □

We note that, with the aid of (2.7) and (2.10), the right-hand side of (4.16) may be computed without recourse to the Clifford product, using *only* the vector operations of multiplication by a scalar, addition and scalar product:

$$\mathbf{U}_{m+1}^J = \mathbf{U}_m^{J+2} + 2(\mathbf{W}\cdot\mathbf{U}_m^{J+1})\mathbf{U}_m^{J+1} - (\mathbf{U}_m^{J+1}\cdot\mathbf{U}_m^{J+1})\mathbf{W} \tag{4.18}$$

where

$$\mathbf{W} := (\mathbf{U}_{m-1}^{J+2})^{-1} - (\mathbf{U}_m^J)^{-1}.$$

The vectors $\mathbf{U}_m^J$ may be arrayed in a table, as in Figure 2, the first two columns of which are initialised using (4.17). Other entries are calculated with the help of (4.16) or (4.18), working from left to right. In this way, the rhombus rules for the q-d algorithm [22] are replaced by a cross rule, as indicated in Fig.4. An instance of the rule (4.16), for $J = 0$, is derived in [7] using ideas from the Berlekamp-Massey algorithm [2,13].

$$\begin{array}{ccccc} & \mathbf{U}_0^0 & & & \\ \mathbf{U}_{-1}^2 & \mathbf{U}_0^1 & \mathbf{U}_1^0 & & \\ \mathbf{U}_{-1}^3 & \mathbf{U}_0^2 & \mathbf{U}_1^1 & \mathbf{U}_2^0 & \\ \mathbf{U}_{-1}^4 & \mathbf{U}_0^3 & \mathbf{U}_1^2 & \mathbf{U}_2^1 & \mathbf{U}_3^0 \\ . & . & . & . & . \quad . \end{array}$$

Figure 2: Part of the $\mathbf{U}$-table

$$\begin{array}{ccc} & \boxed{\mathrm{N}} & \\ \boxed{\mathrm{W}} & \boxed{\mathrm{C}} & \boxed{\mathrm{E}} \\ & \boxed{\mathrm{S}} & \end{array} \equiv \begin{array}{ccc} & \mathbf{U}_m^J & \\ \mathbf{U}_{m-1}^{J+2} & \mathbf{U}_m^{J+1} & \mathbf{U}_{m+1}^J \\ & \mathbf{U}_m^{J+2} & \end{array}$$

Figure 3: Detail of the $\mathbf{U}$-table

The elements $q_m^J, e_m^J, m = 1, 2 \cdots$, in the continued fraction expansion (4.8) may be formed from those entries in the diagonals of the $\mathbf{U}$-table labelled by J and $J+1$. This allows, in principle, the construction of the polynomials in $C\ell_d[z]$ of any vector Padé approximant of order $[l/m]$ with $l \geq m$. It is readily seen that the coefficients of these polynomials are sums of products of vectors or inverted vectors from the $\mathbf{U}$-table. As a simple example we may use (4.2) to demonstrate that the denominator of the $[J+1/2]$ approximant is given by

$$B_4^J(z) = 1 - z(q_1^J + e_1^J + q_2^J) + z^2 q_1^J q_2^J \tag{4.19}$$

which, employing (4.14), is equal to

$$1 - z[(\mathbf{U}_0^J)^{-1}\mathbf{U}_0^{J+1} + (\mathbf{U}_0^{J+1})^{-1}\mathbf{U}_1^J + (\mathbf{U}_1^J)^{-1}\mathbf{U}_0^{J+1}] + z^2[(\mathbf{U}_0^J)^{-1}\mathbf{U}_0^{J+1}(\mathbf{U}_1^J)^{-1}\mathbf{U}_1^{J+1}]. \tag{4.20}$$

We note, in passing, that this form is convenient for the calculation of the denominators of hybrid approximants [7,19]. Such denominators are scalar polynomials

$$\boxed{\mathrm{E}} = \boxed{\mathrm{S}} + \boxed{\mathrm{C}}\left[\boxed{\mathrm{W}}^{-1} - \boxed{\mathrm{N}}^{-1}\right]\boxed{\mathrm{C}}$$

Figure 4: Cross rule relating elements of Fig.3

retaining the nominal degree of the approximant (in this case 2), unlike the form of eqn.(3.12), which would yield a quartic for this example. In order to construct this denominator, we require the scalar part of (4.20), which may be obtained using the result

$$Re(\mathbf{v_1 v_2 v_3 v_4}) = (\mathbf{v_1} \cdot \mathbf{v_2})(\mathbf{v_3} \cdot \mathbf{v_4}) - (\mathbf{v_1} \cdot \mathbf{v_3})(\mathbf{v_2} \cdot \mathbf{v_4}) + (\mathbf{v_1} \cdot \mathbf{v_4})(\mathbf{v_2} \cdot \mathbf{v_3}) \quad (4.21)$$

where $\mathbf{v_i} \in \mathbb{R}^d$, for $i = 1, 2, 3, 4$.

5 Full Tables

The algorithms presented in the previous section may be used to calculate approximants above the main diagonal in the vector Padé table (Fig.1) *c.f.* (3.10). Entries in the lower half of this table may be computed using extensions of these algorithms. To demonstrate this, we proceed as in the scalar case by considering the reciprocal power series to $\mathbf{f}(z)$

$$\mathbf{g}(z) := [\mathbf{f}(z)]^{-1} = \sum_{i=0}^{\infty} \mathbf{d}_i z^i \quad (5.1)$$

which exists if $\mathbf{c}_0 \neq \mathbf{0}$. Using Theorem 4.3, assuming non-degeneracy for $\mathbf{g}(z)$, we may construct the corresponding $\mathbf{U}$-table for $\mathbf{g}(z)$, whose elements are denoted by $\mathbf{U'}_m^J$. We now prove that we may identify entries on the top three diagonals of the $\mathbf{U}$-table of figure 2, excluding boundary elements, with the entries in the same positions of the $[\hat{\mathbf{c}}_0 \mathbf{U}' \mathbf{c}_0]$-table reflected through the diagonal $J = 1$.

Lemma 5.1 *If* $\mathbf{c}_0 \neq 0$ *then*

$$\mathbf{U}_m^J = \hat{\mathbf{c}}_0 \mathbf{U'}_{m'}^{J'} \mathbf{c}_0 \quad (5.2)$$

for

$$J = 0 \quad with \quad m = 1, 2, \cdots, \qquad and \quad J = 1, 2 \quad with \quad m = 0, 1, \cdots.$$

where $J' := 2 - J$, *and* $m' := m + J - 1$.

Proof. Consider the continued fraction corresponding to $\mathbf{g}(z)$ *c.f.* eqn(4.8),

$$\mathbf{C'}^{J'}(z) = \sum_{i=0}^{J'-1} \mathbf{d}_i z^i + \mathbf{d}_{J'} z^{J'} [1 - z{q'}_1^{J'} [1 - z{e'}_1^{J'} [\cdots]^{-1}]^{-1}]^{-1}. \quad (5.3)$$

the n^{th} convergent of which satisfies

$$\mathbf{g}(z) - \mathbf{C'}_n^{J'}(z) = \mathbf{s'}_n^{J'} z^{J'+n} + O(z^{J'+n+1}). \quad (5.4)$$

Using the duality property for vector Padé approximants [section 3], we may infer, from (3.10), that the convergents $[\mathbf{C}_{n+1}^0(z)]^{-1}$ generate the same staircase sequence of approximants — $\{[0/0], [1/0], [1/1], [2/1], \cdots\}$, as the convergents $\mathbf{C'}_n^1(z)$; i.e.

$$\mathbf{C'}_n^1(z) \equiv [\mathbf{C}_{n+1}^0(z)]^{-1} \qquad n = 0, 1, 2 \cdots$$

On multiplying (5.4), with $J' = 1$, from the left by $\mathbf{f}(z)$, and from the right by $\mathbf{C}^0_{n+1}(z)$, we obtain

$$\mathbf{f}(z) - \mathbf{C}^0_{n+1}(z) = \hat{\mathbf{c}}_0 \mathbf{s'}^1_n \mathbf{c}_0 z^{n+1} + O(z^{n+2})$$

assuming $\mathbf{c}_0 \neq \mathbf{0}$. Comparing this result with (4.4) for $J = 0$, we have

$$\mathbf{s}^0_{n+1} = \hat{\mathbf{c}}_0 \mathbf{s'}^1_n \mathbf{c}_0 \qquad n = 0, 1, 2 \cdots . \tag{5.5}$$

For $n = 2m + 1$ we deduce

$$\mathbf{U}^0_{m+1} = \hat{\mathbf{c}}_0 \mathbf{U'}^2_m \mathbf{c}_0 \qquad m = 0, 1, 2, \cdots \tag{5.6}$$

using (4.13), while for $n = 2m$

$$\mathbf{U}^1_m = \hat{\mathbf{c}}_0 \mathbf{U'}^1_m \mathbf{c}_0 \qquad m = 0, 1, 2, \cdots . \tag{5.7}$$

Similarly, the convergents $\mathbf{C'}^0_{n+1}(z)$ generate the same sequence of approximants as $[\mathbf{C}^1_n(z)]^{-1}$. Hence,

$$\mathbf{s}^1_n = \hat{\mathbf{c}}_0 \mathbf{s'}^0_{n+1} \mathbf{c}_0 \qquad n = 0, 1, 2 \cdots . \tag{5.8}$$

For even values of n this is identical to (5.7), whereas for $n = 2m + 1$ we obtain

$$\mathbf{U}^2_m = \hat{\mathbf{c}}_0 \mathbf{U'}^0_{m+1} \mathbf{c}_0 \qquad m = 0, 1, 2, \cdots , \tag{5.9}$$

which completes the proof. □

The equations (5.6), (5.7) and (5.9) constitute the top three diagonals of the $\mathbf{U}$-table, excluding boundary entries, *c.f.* Fig.2. Equation (5.2) may be regarded as defining elements, $\mathbf{U}^J_m$, with negative superscripts which may be arrayed in the full table as depicted in Fig.5.

Since the cross algorithm, eqn(4.16), is symmetric about the diagonal running from the north-west to the south-east, we may apply it in its original form to construct the reflected table. However, in order to obtain the correct value for $\mathbf{U}^0_0$, using the reflected $\mathbf{U}'$-table, we must set $\mathbf{U'}^2_{-1} := -\mathbf{d}_0$, which in turn implies $\mathbf{U'}^0_0 := \infty$. The progressive form of the cross rule now reads as follows :

$$\mathbf{U}^{J+2}_m = \mathbf{U}^J_{m+1} + \mathbf{U}^{J+1}_m [(\mathbf{U}^J_m)^{-1} - (\mathbf{U}^{J+2}_{m-1})^{-1}] \mathbf{U}^{J+1}_m \tag{5.10}$$

with the initialisations :

$$\mathbf{U}^0_0 := \mathbf{c}_0 \quad \mathbf{U}^{-m}_m := \infty \quad \mathbf{U}^{2-m}_{m-1} := \mathbf{h}_m \quad \mathbf{U}^{1+m}_{-1} := \infty \tag{5.11}$$

in which $\mathbf{h}_m := \hat{\mathbf{c}}_0 \mathbf{d}_m \mathbf{c}_0$, for $m = 1, 2, \cdots$. Thus, entries in the full $\mathbf{U}$-table are calculated row by row.

The vectors $\mathbf{h}_m$ may be computed recursively as follows. We first of all construct the real numbers s_i where

$$\sum_{i=0}^{\infty} s_i z^i := \mathbf{f}(z) \cdot \mathbf{f}(z) \tag{5.12}$$

Then we may demonstrate that

$$\mathbf{h}_0 = -\mathbf{c}_0 \tag{5.13}$$

and

$$\mathbf{h}_m = \mathbf{c}_m - \frac{1}{s_0}[2\mathbf{c}_0(\mathbf{c}_0 \cdot \mathbf{c}_m) + \sum_{i=0}^{m-1} s_{m-i}\mathbf{h}_i] \tag{5.14}$$

for $m = 1, 2, \cdots$. Use of (5.10) then enables the full $\mathbf{U}$-table to be calculated using *only* vector operations after the fashion of (4.18).

$$\begin{array}{ccccccc}
 & \mathbf{U}_0^0 & \mathbf{U}_1^{-1} & \mathbf{U}_2^{-2} & \cdot & \cdot \\
\mathbf{U}_{-1}^2 & \mathbf{U}_0^1 & \mathbf{U}_1^0 & \mathbf{U}_2^{-1} & \cdot & \cdot \\
\mathbf{U}_{-1}^3 & \mathbf{U}_0^2 & \mathbf{U}_1^1 & \mathbf{U}_2^0 & \cdot & \cdot \\
\mathbf{U}_{-1}^4 & \mathbf{U}_0^3 & \mathbf{U}_1^2 & \mathbf{U}_2^1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Figure 5: A section of the full $\mathbf{U}$-table

In order to construct the full q-d table, we use (4.14) and Lemma 5.1 to write

$$q'^{J'}_{m'} = \mathbf{c}_0 e_m^{J-1}[\mathbf{c}_0]^{-1} \quad \text{and} \quad e'^{J'}_{m'} = \mathbf{c}_0 q_{m+1}^{J-1}[\mathbf{c}_0]^{-1} \tag{5.15}$$

where $J' = 2 - J$ and $m' = m + J - 1$, which define quantities with negative superscripts. These relations yield the usual results for the scalar (i.e. commutative) case *e.g.* [3,10]. The q-d algorithm then takes the progressive form of (4.11), with the initialisations:

$$\left.\begin{array}{ll} e_m^0 := 0 & e_m^{1-m} := [\mathbf{h}_m]^{-1}\mathbf{h}_{m+1} \\ q_{-m}^{m+1} := 0 & q_1^0 := -[\mathbf{h}_0]^{-1}\mathbf{h}_1 \end{array}\right\} \text{for } m = 1, 2 \cdots \tag{5.16}$$

which also reduce to the usual expressions in the commutative case.

We complete this section by indicating how entries in the lower half of the vector Padé table may be generated from the staircase sequence $[0/k], [0/k+1], [1/k+1], [1/k+2], [2/k+2], \cdots$. They are given by the inverted convergents of (5.3)

$$[\sum_{i=0}^{k} \mathbf{d}_i z^i + \mathbf{d}_{k+1} z^{k+1} \mathbf{c}_0 F_k(z)^{-1} \mathbf{c}_0{}^{-1}]^{-1} \tag{5.17}$$

where

$$F_k(z) := 1 - ze_{k+1}^{-k}[1 - zq_{k+2}^{-k}[1 - ze_{k+2}^{-k}[1 - zq_{k+3}^{-k}[\cdots]^{-1}]^{-1}]^{-1}]^{-1}. \tag{5.18}$$

Again, this corresponds to the usual situation for commuting elements $(d = 1)$ *e.g.* [3].

6 Complex Maclaurin Coefficients

The presentation above is concerned with the application of real Clifford algebras to the rational approximation of functions whose Maclaurin series involve *real* vector coefficients (3.1). If these vectors are *complex* then there is more than one way to proceed, depending on the definition of vector inverse employed, thus generating different types of approximant.

One route is to use the real Clifford algebra, $C\ell_{2d+1}$, to incorporate the Moore-Penrose generalised inverse

$$\mathbf{v}^{-1} := \frac{\mathbf{v}^*}{\mathbf{v} \cdot \mathbf{v}^*} \qquad \mathbf{v} \in \mathbb{C}^d \tag{6.1}$$

where the superscript * denotes complex conjugation. Thus, any non-null vector is invertible. This leads to the construction of generalised inverse Padé approximants (see *e.g.*[5]). Each complex vector $\mathbf{v} = \mathbf{x} + i\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ is associated with the element of $C\ell_{2d+1}$ defined by [6,14]

$$V := \sum_{n=1}^{d} (x_n \mathbf{e}_n + y_n \mathbf{j} \mathbf{e}_{n+d}) \qquad \mathbf{j} := \mathbf{e}_{2d+1} \tag{6.2}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_d)$. In this fashion $\mathbb{C}^d$ is regarded as a real vector space $\mathbb{R}^{2d}$ with basis vectors $\{\mathbf{e}_n, \mathbf{j}\mathbf{e}_{n+d}\}_{n=1}^{d}$; scalar multiplication is by reals only [14]. Complex conjugation of vectors is represented by reversion in the Clifford algebra, while in $\mathbb{R}^{2d}$ it is described by an involution composed of a sequence of reflections.

The identities (2.6) and (2.7) are replaced by

$$U\tilde{V} + V\tilde{U} = 2(\mathbf{u} * \mathbf{v}) \tag{6.3}$$

and

$$U\tilde{V}U = 2(\mathbf{u} * \mathbf{v})U - (\mathbf{u} * \mathbf{u})V \tag{6.4}$$

respectively, where

$$\mathbf{u} * \mathbf{v} := \frac{1}{2}(\mathbf{u} \cdot \mathbf{v}^* + \mathbf{v} \cdot \mathbf{u}^*) \tag{6.5}$$

It follows that (6.1) is represented in the algebra by

$$V^{-1} = \frac{\tilde{V}}{\mathbf{v} * \mathbf{v}}. \tag{6.6}$$

For equivalent forms to (3.11) and (3.12) the reader is referred to [6].

The derivation of the q-d and cross algorithms proceeds as for the case of real vectors with (6.4) and (6.6) being used to compute the right-hand side of (4.16).

The Lipschitz group may be generalised to include products of non-null vectors of the form displayed in (6.2). The corresponding generalisation of (2.12) represents rotations in $\mathbb{R}^{2d}$ leaving invariant the form $\mathbf{u} * \mathbf{v}$ which is proportional to the usual scalar product in $\mathbb{R}^{2d}$.

An alternative approach is simply to use complex Clifford algebras, *i.e.* in (2.3) allow $a_I \in \mathbb{C}\ \ \forall I$, with $\mathbf{v}^{-1}$ given by (2.10) for $\mathbf{v} \in \mathbb{C}^d$, provided $\mathbf{v}$ is non-isotropic — *i.e.* $\mathbf{v} \cdot \mathbf{v} \neq 0$. The identities (2.6), (2.7) hold for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^d$. We strengthen the assumption of non-degenerate corresponding continued fractions ($\boldsymbol{\pi}^J_m \neq 0$) to one of non-isotropy of $\boldsymbol{\pi}^J_m$ for all relevant J, m . This ensures that each α^J_n is invertible, thus allowing the equivalence transformation (3.14) to be performed.

Lemma 6.1 *Given* $n, J \geq 0,$ *then*

$$\mathbf{s}^J_0, \mathbf{s}^J_1, \cdots \mathbf{s}^J_n \quad \text{are invertible} \quad \Longleftrightarrow \quad \boldsymbol{\pi}^J_1, \boldsymbol{\pi}^J_2, \cdots \boldsymbol{\pi}^J_{n+1} \quad \text{are invertible.} \tag{6.7}$$

Proof From the last equation of section 3 of [20] we obtain

$$\mathbf{f}(z) - \mathbf{C}^J_n(z) = (-1)^n[\boldsymbol{\pi}^J_1 \cdots \boldsymbol{\pi}^J_n \boldsymbol{\pi}^J_{n+1} \boldsymbol{\pi}^J_n \cdots \boldsymbol{\pi}^J_1]^{-1} z^{J+n} + O(z^{J+n+1}).$$

Hence, provided the required inverses exist,

$$\mathbf{s}^J_n = [\alpha^J_n \boldsymbol{\pi}^J_{n+1} \overline{\alpha^J_n}]^{-1} \quad n = 0, 1, 2, \cdots \tag{6.8}$$

with $\alpha^J_0 := 1$. The result then follows. $\square$

Equation (6.8) is consistent with (3.17) and (4.5). We also note that the condition for a Padé table to be normal in the scalar case, *i.e.* all $s^J_n \neq 0$, may be replaced in the vector case by the requirement that each of the inverses $[\mathbf{s}^J_n]^{-1}$ exists. However, if (6.1) is used instead of (2.10) then we may retain the less stringent constraint $\mathbf{s}^J_n \neq \mathbf{0}$.

The derivations of the q-d and cross algorithms given in sections 4 and 5 are valid in the complex case. Furthermore, since each $\mathbf{U}^J_m$ is invertible under our assumption of non-isotropy, these algorithms may be implemented using (2.7) and (2.10).

The Lipschitz group is extended to include products of non-isotropic complex vectors. The transformation (2.12) represents a complex rotation in $\mathbb{C}^d$, leaving invariant the bilinear quadratic form $\mathbf{u} \cdot \mathbf{v}$, for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^d$. Given that the numerator and denominator polynomials of the continued fraction (3.7), regarded as functions of $z \in \mathbb{C}$, belong to the extended Lipschitz group — see [18], an argument similar to that preceding Theorem 3.1 shows that the transformed polynomials are also members of this group. Finally, (3.11) and (3.12) hold with $\mathbb{R}$ replaced by $\mathbb{C}$.

In the next section we consider vector-valued meromorphic funtions satisfying certain conditions, and show that the problem of degeneracy using the vector inverse (2.10) for complex vectors does not occur for large enough values of J. We take advantage of convergence results [21] which are readily obtained using (2.10). A corresponding study could be undertaken based on the inverse (6.1) using theorems proved by Graves-Morris and Saff [8,9].

7 Vector-valued Meromorphic Functions

In section 5 we described the possible construction of the $\mathbf{U}$-table for functions whose Maclaurin series are known. Now we consider the behaviour of this table (and of the related q-d table) for functions of a particular type — vector-valued meromorphic functions. These are the objects of interest in the application of vector Padé approximants to matrix iterative processes, see e.g. [4]. Attention is focussed on the behaviour along rows of the vector Padé table *i.e.* in the large J behaviour of $\mathbf{C}_n^J$ *c.f.*(3.10).

We adopt the definition of inverse given by (2.10) for complex vectors, and consider those functions involved in the convergence results of [21], some of which are quoted in Theorem 7.1 below.

We define

$$\mathbf{f}(z) := \frac{\mathbf{g}(z)}{R_M(z)} \qquad z \in \mathbb{C} \tag{7.1}$$

where

$$R_0(z) := 1 \qquad R_m(z) := \prod_{k=1}^{m}(z - z_k) \quad \text{for } z_k \in \mathbb{C} \quad \text{and} \quad m = 1, 2, \cdots, M \tag{7.2}$$

such that

$$0 < |z_1| \leq |z_2| \leq \cdots \leq |z_M| < \rho \tag{7.3}$$

counting multiplicity. Each component of $\mathbf{g}(z)$ is an analytic function for $z \in D_\rho := \{z \in \mathbb{C} : |z| < \rho\}$, and we assume that

$$\mathbf{g}(z_k) \cdot \mathbf{g}(z_k) \neq 0 \quad \text{for} \quad k = 1, 2, \cdots, M. \tag{7.4}$$

Theorem 7.1 *Given a vector-valued function satisfying 7.1-7.4, then, for sufficiently large l, the vector Padé approximant to* $\mathbf{f}(z)$

$$[l/M](z) = p^{[l/M]}(z)[q^{[l/M]}]^{-1}$$

exists and

$$\lim_{l\to\infty}[l/M](z) = \mathbf{f}(z)$$

the convergence being uniform in compact subsets of $D_\rho^- := D_\rho - \cup_{k=1}^M z_k$. *Furthermore, if each* $q^{[l/M]}(z)$ *is monic, then*

$$\lim_{l\to\infty} q^{[l/M]}(z) = R_M(z)$$

the convergence being uniform in any bounded subset E *of the complex plane.*
The reader is referred to [21] for a proof.

Theorem 7.2 *Given a vector-valued function satisfying 7.1-7.4, then*

$$\lim_{J\to\infty}\{[z_{m+1}]^J \mathbf{U}_m^J\} = -\mathbf{r}_{m+1}\frac{[D_m(z_{m+1})]^2}{[z_{m+1}]^{2m+1}} \quad , m = 0, 1, \cdots, M-1 \tag{7.5}$$

if $|z_m| < |z_{m+1}| < |z_{m+2}|$, *where the principal part of* $\mathbf{f}(z)$ *at* z_{m+1} *is*

$$\frac{\mathbf{r}_{m+1}}{z - z_{m+1}}$$

and

$$D_0(z) := 1 \quad , \quad D_m(z) := \prod_{k=1}^m (1 - z/z_k) \quad m > 0$$

with $z_0 := 0$ *and* $z_{M+1} := \rho$.

Proof If $|z_m| < |z_{m+1}| < |z_{m+2}|$, then from Theorem (7.1) it follows that, for sufficiently large l, the $[l/m]$ vector Padé approximant to $\mathbf{f}(z)$,as defined above, exists and that the denominator polynomial $q^{[l/m]}(z)$ tends to the scalar-valued function $R_m(z)$ uniformly, as $l \to \infty$, in any bounded subset of the complex plane. From [21] we may state the generalised Hermite error formula for vector Padé approximants as follows :

$$\mathbf{f}(z) - [l/m](z) = \frac{z^{l+m+1}}{2\pi i R_m(z)} \oint_{|v|=\sigma} \frac{\mathbf{g}_m(v) q^{[l/m]}(v) dv}{v^{l+m+1}(v - z)} [q^{[l/m]}(z)]^{-1} \tag{7.6}$$

where $\mathbf{g}_m(v) := R_m(v)\mathbf{f}(v)$ and $|z_m| < \sigma < |z_{m+1}|$. From (3.10), (4.7) and (4.13) we conclude that

$$\mathbf{U}_m^J = \frac{1}{2\pi i}\oint_{|v|=\sigma} \frac{\mathbf{g}_m(v) q^{[l/m]}(v) dv}{v^{J+2m+1}} \cdot \frac{[q^{[l/m]}(0)]^{-1}}{R_m(0)} \tag{7.7}$$

for $J := l - m + 1$. Expanding the contour in (7.7) to include the simple pole at z_{m+1}, $\mathbf{U}_m^J$ is given by

$$\frac{1}{2\pi i}\oint_{|v|=\sigma'} \frac{\mathbf{g}_m(v) q^{[l/m]}(v) dv}{v^{J+2m+1}} \frac{[q^{[l/m]}(0)]^{-1}}{R_m(0)} - \frac{\mathbf{r}_{m+1} R_m(z_{m+1})}{[z_{m+1}]^{J+2m+1} R_m(0)} q^{[l/m]}(z_{m+1})[q^{[l/m]}(0)]^{-1} \tag{7.8}$$

for $|z_{m+1}| < \sigma' < |z_{m+2}|$. We note that $[q^{[l/m]}(0)]^{-1}$ exists and is bounded for l large enough [21]. The integrand in (7.8) is also bounded on the contour $\Gamma' := \{v \in \mathbb{C} : |v| = \sigma'\}$. To see this, we use (2.5) to observe that :

$$|\mathbf{g}_m(v)q^{[l/m]}(v)| \leq K_d|\mathbf{g}_m(v)||q^{[l/m]}(v)|.$$

Since $q^{[l/m]}(v) \to R_m(v)$ uniformly on bounded subsets of $\mathbb{C}$, then the denominator polynomial is bounded on Γ' for sufficiently large l. Finally, the definition of $\mathbf{g}_m(v)$ ensures that each of its component functions is continuous and therefore bounded on Γ', thus proving that the integral in (7.8) is $O(\sigma'^{-J})$. Hence,

$$[z_{m+1}]^J\mathbf{U}_m^J \to -\frac{\mathbf{r}_{m+1}}{[z_{m+1}]^{2m+1}}\left[\frac{R_m(z_{m+1})}{R_m(0)}\right]^2 \quad \text{as } J \to \infty \quad \text{for } m = 0, 1, \cdots, M-1$$

where $z_0 := 0$ and $z_{M+1} := \rho$. Result (7.5) then follows. □

Corollary 7.3 *Given a vector-valued meromorphic function of the type considered above then :*

$$\lim_{J\to\infty} q_m^J = \frac{1}{z_m} \tag{7.9}$$

for $|z_{m-1}| < |z_m| < |z_{m+1}|$ *and* $m = 1, 2\cdots, M$ *— i.e. the bivector part of* q_m^J *vanishes for large* J*; and*

$$\lim_{J\to\infty}[e_m^J(\frac{z_{m+1}}{z_m})^J] = z_m[\mathbf{r}_m]^{-1}\mathbf{r}_{m+1}[\frac{z_m}{z_{m+1}}]^{2m+1}\left[\frac{D_m(z_{m+1})}{D_{m-1}(z_m)}\right]^2 \tag{7.10}$$

for $|z_{m-1}| < |z_m| < |z_{m+1}| < |z_{m+2}|$ *and* $m = 1, 2\cdots, M-1$.
Hence,

$$e_m^J = O(|\frac{z_m}{z_{m+1}}|^J). \tag{7.11}$$

The proof follows from Theorem 7.2. We note that (7.4) implies the existence of $[\mathbf{r}_m]^{-1}$.

If each component of $\mathbf{g}(z)$ in (7.1) is a polynomial of maximum degree L, then $\mathbf{U}_m^J = \mathbf{0}$ for $m \geq M$ and $J > L - m$, since each approximant $\mathbf{C}_{2m}^J$ is exact for these values of J and m.

In the context of matrix iterative methods the vector residue $\mathbf{r}_{m+1}$ is an eigenvector of the iteration matrix corresponding to the eigenvalue $1/z_{m+1}$ [4].

Theorem 7.4 *If a vector-valued function satisfying (7.1)-(7.4) has poles of distinct moduli then there exists a number* J_M *such that, in principle, the* $\mathbf{U}$*-table may be constructed for* $J \geq J_M$ *and* $0 \leq m < M$.

Proof We begin by showing that for sufficiently large J (i) the denominator polynomials of the even convergents of (3.7), $q_{2m}^J(z)$, are monic and are of exact degree m, while (ii) the vectors $\boldsymbol{\pi}_k^J$, $k = 1, 2, \cdots, 2m$ are non-isotropic.

By eliminating q^J_{2m-1} and q^J_{2m+1} from the recurrence relations (3.9) for $n = 2m, 2m+1, 2m+2$, we obtain

$$q^J_{2m+2}(z) = q^J_{2m}(z)\{\boldsymbol{\pi}^J_{2m+1}\boldsymbol{\pi}^J_{2m+2}+z[(\boldsymbol{\pi}^J_{2m})^{-1}\boldsymbol{\pi}^J_{2m+2}+1]\}-z^2 q^J_{2m-2}(z)(\boldsymbol{\pi}^J_{2m})^{-1}\boldsymbol{\pi}^J_{2m+2} \tag{7.12}$$

for $m = 1, 2, \cdots$. The initialisations are given by

$$q^J_0(z) := 1 \quad \text{and} \quad q^J_2(z) := z + \boldsymbol{\pi}^J_1\boldsymbol{\pi}^J_2. \tag{7.13}$$

It is clear that (i) holds for $m = 1$. Since $|z_1| < |z_2|$ Theorem 7.1 implies that $q^J_2(z) \to (z - z_1)$ as $J \to \infty$. Hence,

$$\boldsymbol{\pi}^J_1\boldsymbol{\pi}^J_2 \to -z_1 \quad \text{as} \quad J \to \infty.$$

On applying reversion to each side and then multiplying the resulting expressions, we observe that

$$(\boldsymbol{\pi}^J_1)^2(\boldsymbol{\pi}^J_2)^2 \to (z_1)^2 \neq 0 \quad as \quad J \to \infty.$$

Therefore, there exists a non-zero integer J_1, such that each of the vectors $\boldsymbol{\pi}^J_1$ and $\boldsymbol{\pi}^J_2$ is non-isotropic for $J \geq J_1$, thus ensuring the validity of (ii) for $m = 1$.

We now assume that (i) and (ii) hold for $m = 1, 2, \cdots, k$ and $J \geq J_k$. If $J \geq J_k$, then $\boldsymbol{\pi}^J_{2k}$ is invertible, allowing the construction of $q^J_{2k+2}(z)$ from (7.12). It is readily seen that the highest power of this polynomial is $k+1$ with coefficient unity. Since $|z_{k+1}| < |z_{k+2}|$ for $k = 1, 2, \cdots M-1$ ($z_{M+1} := \rho$) Theorem 7.1 implies that

$$q^J_{2k+2}(z) \to R_{k+1}(z) \quad \text{as} \quad J \to \infty.$$

However, as noted in section 3, $q^J_{2k+2}(0) = \boldsymbol{\pi}^J_1\boldsymbol{\pi}^J_2 \cdots \boldsymbol{\pi}^J_{2k+1}\boldsymbol{\pi}^J_{2k+2}$. Hence,

$$\boldsymbol{\pi}^J_1\boldsymbol{\pi}^J_2 \cdots \boldsymbol{\pi}^J_{2k+1}\boldsymbol{\pi}^J_{2k+2} \to \prod_{i=1}^{k+1}(-z_i) \quad as \quad J \to \infty. \tag{7.14}$$

Therefore, there exists $J_{k+1} \geq J_k$, such that the product $\boldsymbol{\pi}^J_{2k+1}\boldsymbol{\pi}^J_{2k+2}$ is invertible for $J \geq J_{k+1}$. That is each of the vectors, $\boldsymbol{\pi}^J_{2k+1}$ and $\boldsymbol{\pi}^J_{2k+2}$, is non-isotropic for $J \geq J_{k+1}$.

Thus we have shown the existence of an integer J_M such that, for $J \geq J_M$, each of the vectors $\boldsymbol{\pi}^J_1, \boldsymbol{\pi}^J_2, \cdots, \boldsymbol{\pi}^J_{2M-1}, \boldsymbol{\pi}^J_{2M}$ is non-isotropic and finite. It then follows from Lemma 6.1 and (4.13) that $\mathbf{U}^J_0, \mathbf{U}^J_1, \cdots, \mathbf{U}^J_{M-1}$ are non-isotropic and finite for $J \geq J_M$; that is, the section of the $\mathbf{U}$-table corresponding to $J \geq J_M$ and $0 \leq m < M$, may be constructed using Theorem 4.3 . □

We may now state

Theorem 7.5 *Given a vector-valued meromorphic function of the type considered above, for which the poles have distinct moduli, then the asymptotic behaviour of the* $\boldsymbol{\pi}_n^J$ *and the* α_n^J *are determined by*

$$\lim_{J\to\infty} \alpha_{2m}^J = (-1)^m z_1 z_2 \cdots z_m \tag{7.15}$$

$$\lim_{J\to\infty} \boldsymbol{\pi}_{2m-1}^J \boldsymbol{\pi}_{2m}^J = -z_m \tag{7.16}$$

$$\lim_{J\to\infty} \{{z_m}^J \boldsymbol{\pi}_{2m}^J\} = \begin{cases} \mathbf{r}_1 & m=1 \\ \mathbf{r}_m\Big[\prod_{i=1}^{m-1}(1-\frac{z_i}{z_m})^2\Big] & m>1 \end{cases} \tag{7.17}$$

for $m = 1, 2, \cdots, M$ *.*

Proof Statement (7.15) is equivalent to (7.14), while (7.16) follows from (7.15) and the observation

$$\boldsymbol{\pi}_{2m-1}^J \boldsymbol{\pi}_{2m}^J = [\alpha_{2m-2}^J]^{-1}\alpha_{2m}^J.$$

Note that (7.15) implies the existence of the inverse for large enough J. To prove (7.17) we write $[z_m]^J \boldsymbol{\pi}_{2m}^J$ as

$$[z_m]^J[\boldsymbol{\pi}_{2m-1}^J]^{-1}[\boldsymbol{\pi}_{2m-1}^J \boldsymbol{\pi}_{2m}^J] = \overline{\alpha_{2m-2}^J}\{[z_m]^J \mathbf{U}_{m-1}^J\}\alpha_{2m-2}^J[\boldsymbol{\pi}_{2m-1}^J \boldsymbol{\pi}_{2m}^J]$$

using (6.8) and (4.13). Theorem 7.2 together with (7.15) and (7.16) imply that, as $J \to \infty$, the right-hand side tends to

$$\mathbf{r}_m \prod_{i=1}^{m-1} (1 - \frac{z_i}{z_m})^2$$

for $m > 1$, as required. For $m = 1$, the above argument yields

$$[z_1]^J \boldsymbol{\pi}_2^J = [z_1]^J \mathbf{U}_0^J \alpha_2^J.$$

Then, using Theorem 7.2 and (7.15) we may conclude that the right-hand side tends to $\mathbf{r}_1$ as $J \to \infty$. □

Note that (7.15) is consistent with (3.19) and (7.9), although the derivation of the latter does not depend on *all* the eigenvalues having distinct moduli. The limit (7.15), for large J, implies that the (perhaps complex) rotations indicated by (3.18) tend to the identity for even i, and to a reflection through the origin for odd i. We also point out that

$$\boldsymbol{\pi}_{2m}^J = O(\frac{1}{|z_m|^J}) \quad , \quad \boldsymbol{\pi}_{2m-1}^J = O(|z_m|^J)$$

i.e. $\boldsymbol{\pi}_1^J, \boldsymbol{\pi}_2^J, \boldsymbol{\pi}_3^J, \cdots$ alternate between large and small values, for $|z_m| \neq 1$, as $J \to \infty$.

For an iteration matrix with eigenvalues of distinct moduli (7.5) holds for $m = 0, 1, \cdots, M-1$; each column of the $\mathbf{U}$-table tends to an eigenvector corresponding to an eigenvalue (in decreasing order of modulus from left to right) of the iteration matrix.

We now comment on a computational aspect of the implementation of the cross algorithm. Theorem (7.2) implies that

$$\mathbf{U}_m^J \simeq -\frac{\mathbf{r}_{m+1}}{[z_{m+1}]^{J+2m+1}}[D_m(z_{m+1})]^2 \qquad \text{as } J \to \infty \tag{7.18}$$

i.e.

$$\mathbf{U}_m^J = O(\frac{1}{|z_{m+1}|^J}).$$

Therefore, cancellation between quantities of similar order may occur in the computation of the right-hand side of (4.16) thus leading to numerical instability in finite arithmetic using the column by column implementation of the cross rule — just as for the q-d algorithm in the scalar case [10]. However, the progressive form does not suffer from this defect. Nevertheless, there is still the problem of computational overflow/underflow made apparent in (7.18). We present an attempt to overcome this, illustrating the approach using real vectors and the Euclidean norm. The extension to complex vectors and other norms is fairly straightforward.

If we label the unit vector of $\mathbf{U}_m^J$ by $\mathbf{u}_m^J$, then (4.16), after division by $|\mathbf{U}_m^{J+1}|$, may be written as

$$\beta_{m+1}^J \mathbf{u}_{m+1}^J = \alpha_{m+1}^{J+1}\mathbf{u}_m^{J+2} + \mathbf{u}_m^{J+1}\{\beta_m^{J+1}\mathbf{u}_{m-1}^{J+2} - \alpha_{m+1}^J \mathbf{u}_m^J\}\mathbf{u}_m^{J+1}$$

where

$$\alpha_m^J := |q_m^J| \quad \text{and} \quad \beta_m^J := |e_m^J|.$$

The cross algorithm now takes the form:

Initialisation:

$for \;\; J = 0, 1, \cdots J_{max}$

$$\beta_0^{J+1} := 0 \quad , \quad \alpha_1^J := \frac{|\mathbf{c}_{J+1}|}{|\mathbf{c}_J|} \quad , \quad \mathbf{u}_0^J := \frac{\mathbf{c}_J}{|\mathbf{c}_J|} \tag{7.19}$$

end for.

Iteration:

$for \;\; m = 1, 2, \cdots M-1$

$for \;\; J = 0, 1, \cdots J_{max} - 2m$

$$\left.\begin{aligned} \mathbf{W} &:= \beta_{m-1}^{J+1}\mathbf{u}_{m-2}^{J+2} - \alpha_m^J \mathbf{u}_{m-1}^J \\ \beta_m^J \mathbf{u}_m^J &= \alpha_m^{J+1}\mathbf{u}_{m-1}^{J+2} + 2(\mathbf{W}\cdot\mathbf{u}_{m-1}^{J+1})\mathbf{u}_{m-1}^{J+1} - \mathbf{W} \end{aligned}\right\} \tag{7.20a}$$

end for

$$\alpha_{m+1}^J := \alpha_m^{J+1}\beta_m^{J+1}/\beta_m^J \tag{7.20b}$$

end for $\square$

In this formulation the vectors $\mathbf{u}_{-1}^J$ are assumed to be arbitrary but finite.

The elements in the continued fraction (4.8) may be obtained from:

$$q_m^J = \alpha_m^J \mathbf{u}_{m-1}^J \mathbf{u}_{m-1}^{J+1} \qquad e_m^J = \beta_m^J \mathbf{u}_{m-1}^{J+1} \mathbf{u}_m^J \tag{7.21}$$

If we denote a unit vector of $\mathbf{r}_i$ by $\mathbf{v}_i$, $i = 1, \cdots, M$ then, for the particular conditions of Theorem (7.2) and its Corollary (7.3), we have as $J \to \infty$

$$\left.\begin{aligned} \mathbf{u}_m^J &\to \pm\mathbf{v}_{m+1} \\ \alpha_m^J \mathbf{u}_{m-1}^J \cdot \mathbf{u}_{m-1}^{J+1} &\to z_m \\ \beta_m^J &\to 0 \end{aligned}\right\} m = 0, 1, \cdots, M-1 \tag{7.22}$$

We note that the first of these limits implies that the scalar product in the second tends to ± 1 as $J \to \infty$.

The progressive algorithm may be implemented in a similar fashion.

In the context of matrix iteration $\mathbf{c}_J$ is the J^{th} power of a matrix (here denoted by A) acting on an initial vector $\mathbf{c}_0$. It is in this sense that the cross algorithm affords a generalisation of the power method for calculating eigenvalues and their eigenvectors other than the dominant one. We emphasise that *only the vector iterates* $\mathbf{u}_0^J$ *and the* α_1^J are required in (7.20). These values are obtained from the relations

$$\mathbf{u}_0^0 := \frac{\mathbf{c}_0}{|\mathbf{c}_0|} \quad , \quad \alpha_1^J \mathbf{u}_0^{J+1} := A\mathbf{u}_0^J. \tag{7.23}$$

Example 7.6 As a simple illustration we consider the following matrix and initial vector

$$A := \tfrac{1}{6}\begin{bmatrix} 22 & -8 & 12 \\ 53 & -25 & 42 \\ 22 & -14 & 24 \end{bmatrix} \quad , \qquad \mathbf{c}_0 := \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{7.24}$$

The exact eigenvalues and associated unit eigenvectors are

$$\begin{array}{ccc} \lambda_1 = 2 & \lambda_2 = 1 & \lambda_3 = 0.5 \\ \mathbf{v}_1 = \tfrac{1}{\sqrt{21}}\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} & \mathbf{v}_2 = \tfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} & \mathbf{v}_3 = \tfrac{1}{\sqrt{13}}\begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix} \end{array} \tag{7.25}$$

The row implementation of the cross algorithm was coded first and the numerical instability discussed above was observed — *e.g.* it was found that the accuracy of

α_2^J is increasingly affected by round-off for values of J greater than about 15. The computations were carried out on a SUN sparcstation using Fortran77 in double precision. We next programmed the progressive form using equation (5.14) to calculate the vectors $\mathbf{h}_m$ from the iterates $\mathbf{c}_m := A\mathbf{c}_{m-1}$: we ignored scaling in these operations since our immediate aim is to demonstrate the use of the cross rule in producing the behaviour of (7.22). These vectors allow us to initialise as follows:

$$\beta_0^m := 0, \quad \alpha_1^0 := \frac{|\mathbf{h}_1|}{|\mathbf{h}_0|}, \quad \alpha_{m+1}^{-m}; = 0, \quad \beta_m^{1-m} := \frac{|\mathbf{h}_{m+1}|}{|\mathbf{h}_m|}$$

and

$$\mathbf{u}_0^0 := \frac{\mathbf{c}_0}{|\mathbf{c}_0|} = -\frac{\mathbf{h}_0}{|\mathbf{h}_0|}, \quad \mathbf{u}_{m-1}^{2-m} := \frac{\mathbf{h}_m}{|\mathbf{h}_m|}$$

for $m = 1, 2, \cdots$. The m and J iterations in (7.20) are interchanged so that the cross rule may be used to compute elements row by row. Equation (7.20a) enables α_m^{J+1} and $\mathbf{u}_{m-1}^{J+2}$ to be calculated, while (7.20b) furnishes β_m^{J+1} during each iteration. The results are shown in Tables 1-3 . For large values of J we see that this implementation is numerically more stable than the one first attempted. Use is made of the observations prior to the statement of Theorem 7.4 concerning the third and later columns of the β-table — *i.e.* $\beta_3^J = 0$ for $J \geq 0$. It was also noted that the scalar product used to construct the entries in Table 1 had little effect on α_m^J, regarded on its own, as an estimator for λ_{m+1} for the larger values of J used. Finally, it is clear from the tables that the results are consistent with the behaviour described in (7.22).

Remarks Given a vector-valued power series (3.1), the elements of its corresponding continued fraction expansion (4.8) may be evaluated using the $\mathbf{U}$-table and the relations (4.14). Use of the row form of the cross algorithm to compute $\mathbf{U}_m^J$ (4.16,4.18) is adequate for small J. However, if the power series is that of a vector-valued meromorphic function satisfying (7.1-4), then, for large values of J, exact arithmetic is required for the implementation of (7.20) or, if finite arithmetic is used, resort to the progressive form yields better numerical stability at the cost of greater memory and more computation. In either case, the continued fraction elements q_m^J, e_m^J, are then furnished by (7.21).

As with the scalar q-d algorithm, we note that the implementation of the progressive form of the cross rule in finite arithmetic would be greatly improved, and less memory required, if the entries in the top diagonals, $J = 0, 1$, of the $\mathbf{U}$-table in Fig.2 were known accurately. Other work would involve pursuing the parallel with the scalar case: *e.g.* analysing the rates of convergence of the limits in (7.22), as well as investigating vector-valued functions having poles with equal moduli.

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$k\backslash m$	0	1	2
1	7.11111	0.00000	0.00000
2	2.26909	0.08820	0.00095
3	2.01789	0.30530	1.17680
4	1.98652	0.84967	0.66382
5	1.98813	1.18338	0.32849
6	1.99284	1.10600	0.40116
7	1.99612	1.05260	0.45128
8	1.99799	1.02596	0.47605
9	1.99898	1.01287	0.48815
10	1.99948	1.00641	0.49411
11	1.99974	1.00320	0.49706
12	1.99987	1.00160	0.49853
13	1.99993	1.00080	0.49927
14	1.99997	1.00040	0.49963
15	1.99998	1.00020	0.49982
20	2.00000	1.00001	0.49999
25	2.00000	1.00000	0.50000
λ_{m+1}	2.00000	1.00000	0.50000

Table 1: Estimates of the eigenvalues of the matrix in Example 7.6 given by $\alpha_m^{k-m}\mathbf{u}_{m-1}^{k-m+1} \cdot \mathbf{u}_{m-1}^{k-m}$

$k\backslash m$	1	2	3
1	0.581D+01	0.576D+01	0.576D+01
2	0.261D+00	0.215D+00	0.116D+01
3	0.557D-01	0.623D+00	0.000D+00
4	0.309D-01	0.576D+00	0.000D+00
5	0.189D-01	0.213D+00	0.000D+00
6	0.105D-01	0.796D-01	0.000D+00
7	0.554D-02	0.342D-01	0.000D+00
8	0.285D-02	0.159D-01	0.000D+00
9	0.144D-02	0.767D-02	0.000D+00
10	0.726D-03	0.376D-02	0.000D+00
11	0.364D-03	0.186D-02	0.000D+00
12	0.182D-03	0.928D-03	0.000D+00
13	0.913D-04	0.463D-03	0.000D+00
14	0.457D-04	0.231D-03	0.000D+00
15	0.228D-04	0.116D-03	0.000D+00
20	0.714D-06	0.361D-05	0.000D+00
25	0.223D-07	0.113D-06	0.000D+00

Table 2: Part of the β-table for Example 7.6 — β_m^{k-m}

$k\backslash m$	0	1	2
0	0.57735 0.57735 0.57735	0.00000 0.00000 0.00000	0.00000 0.00000 0.00000
1	0.32004 0.86164 0.39389	0.13325 -0.98925 -0.06027	-0.55381 0.77791 -0.29690
2	0.35689 0.87600 0.32444	-0.38884 -0.91988 -0.05116	-0.45109 0.31652 0.83447
3	0.39060 0.87751 0.27823	-0.65264 -0.52442 0.54685	-0.59098 0.00740 0.80665
4	0.41185 0.87619 0.25034	-0.56050 0.08788 0.82348	-0.61622 -0.69918 0.36252
5	0.42370 0.87482 0.23487	-0.47199 0.29042 0.83240	-0.27945 -0.92843 -0.24479
6	0.42996 0.87393 0.22670	-0.43649 0.35787 0.82547	-0.11654 -0.89119 -0.43841
7	0.43317 0.87342 0.22250	-0.42147 0.38494 0.82109	-0.05298 -0.86208 -0.50400
8	0.43479 0.87315 0.22037	-0.41463 0.39704 0.81880	-0.02528 -0.84702 -0.53096
9	0.43561 0.87301 0.21930	-0.41139 0.40275 0.81765	-0.01235 -0.83951 -0.54320
10	0.43602 0.87294 0.21876	-0.40980 0.40553 0.81707	-0.00611 -0.83577 -0.54904
11	0.43623 0.87291 0.21849	-0.40902 0.40689 0.81679	-0.00304 -0.83391 -0.55189
12	0.43633 0.87289 0.21835	-0.40863 0.40757 0.81664	-0.00151 -0.83298 -0.55330
13	0.43638 0.87288 0.21829	-0.40844 0.40791 0.81657	-0.00076 -0.83251 -0.55400
14	0.43641 0.87288 0.21825	-0.40834 0.40808 0.81653	-0.00038 -0.83228 -0.55435
15	0.43642 0.87287 0.21823	-0.40830 0.40816 0.81651	-0.00019 -0.83217 -0.55453
20	0.43644 0.87287 0.21822	-0.40825 0.40825 0.81650	-0.00001 -0.83205 -0.55469
25	0.43644 0.87287 0.21822	-0.40825 0.40825 0.81650	0.00000 -0.83205 -0.55470
$\mathbf{v}_{m+1}$	0.43644 0.87287 0.21822	0.40825 -0.40825 -0.81650	0.00000 -0.83205 -0.55470

Table 3: Estimates of unit eigenvectors of the matrix in Example 7.6 given by $\mathbf{u}_m^{k-m}$